

## Geometry of Hilbert Space

### Inner Product Spaces.

Let us consider two nonzero vectors

$$x = (x(1), x(2)) \text{ and } y = (y(1), y(2)) \in \mathbb{R}^2.$$

Then  $x \cdot y = x(1)y(1) + x(2)y(2)$ ,

and if  $\theta$  is the angle between  $x$  and  $y$ , ~~not letting~~ then

$$\theta = \arccos\left(\frac{x \cdot y}{\|x\|_2 \cdot \|y\|_2}\right), \quad 0 \leq \theta \leq \pi,$$

where  $\|x\|_2$  denotes the Euclidean norm  $(x(1)^2 + x(2)^2)^{\frac{1}{2}}$  of  $x$ .

We shall generalize the above concept to  $\mathbb{R}^n$ .

Definition. Let  $X$  be a linear space over  $K$ . An inner product on  $X$  is a function  $\langle , \rangle : X \times X \rightarrow K$  such that for all  $x, y, z \in X$  and  $k \in K$ ,

(i)  $\langle x, x \rangle \geq 0$ ,  $\langle x, x \rangle = 0$  iff  $x = 0$  (positive-definiteness)

(ii)  $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

$$\langle kx, y \rangle = k \langle x, y \rangle \text{ (linearity in the first variable)}$$

(iii)  $\langle y, x \rangle = \overline{\langle x, y \rangle}$  (conjugate symmetry).

An inner product space (ips) is a linear space with an inner product on it.

From (ii) and (iii), it follows that inner product is conjugate linear in the second ~~variable~~ variable. Indeed, the following holds.

$$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

$$\text{and } \langle x, ky \rangle = \bar{k} \langle x, y \rangle \quad \forall x, y, z \in X, k \in K.$$

$\bar{k} \rightarrow$  complex conjugate of  $k \in \bar{K}$ .

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Since we wish to allow  $K$  to be  $\mathbb{R}$  or  $\mathbb{C}$ , we shall state our results for  $K = \mathbb{C}$  with the understanding that the bar over  $k \in K$  and all terms involving  $i$  are to be dropped to obtain the corresponding result for the case  $K = \mathbb{R}$ .

Example. (i) For  $x, y \in \mathbb{K}^n$ ,

HW

$$\langle x, y \rangle = \sum_{j=1}^n x(j) \overline{y(j)},$$

$$(ii) \text{ for } x, y \in l^p, \quad \langle x, y \rangle = \sum_{j=1}^{\infty} x(j) \overline{y(j)};$$

(iii) and for  $x, y \in L^2(E)$ , where  $E$  is a measurable subset of  $\mathbb{R}$ ,

$$\langle x, y \rangle = \int_E xy dm$$

are all inner products.

Hint: Use Holder's and Schwarz's inequality and Minkowski inequality.

Lemma. Let  $\langle , \rangle$  be an inner product on a linear space  $X$ . Then the following hold:

(i) Polarization Identity: for all  $x, y \in X$ ,

$$\begin{aligned} \langle x, y \rangle &= \frac{1}{4} \left[ \langle x+y, x+y \rangle - \langle x-y, x-y \rangle + i \langle x+iy, x+iy \rangle \right. \\ &\quad \left. - i \langle x-iy, x-iy \rangle \right]; \end{aligned}$$

and

(ii)  $x \in X$  is the zero element iff  $\langle x, y \rangle = 0 \ \forall y \in X$ .

Proof. (i) Since  $\langle , \rangle$  is linear in the first variable and conjugate linear in the second variable, then the expansion of on the r.h.s lead to  $\langle x, y \rangle$ .

(i) If  $x=0$ , then for all  $y \in Y$ ,

$$\langle 0, y \rangle = \langle 0+y, y \rangle = \langle 0, y \rangle + \langle y, y \rangle$$

so that  $\langle y, y \rangle = 0$ .

conversely, if  $\langle x, y \rangle = 0$  for all  $y \in X$ , let  $y=x$ .

Then  $\langle x, x \rangle = 0$ . Then by the positive-definiteness of an inner product, we have  $x=0$ .

The lemma is proved.

Remark. We know that a norm on a linear space  $X$  is a function  $\| \cdot \| : X \rightarrow K$  such that for all  $x, y \in X, k \in K$ ,

(i)  $\|x\| \geq 0$ , and  $\|x\| = 0 \iff x = 0$ ;

(ii)  $\|x+y\| \leq \|x\| + \|y\|$ ; and

(iii)  $\|kx\| = |k| \|x\|$ .

The conditions of norms and inner product warrant a close comparison.

The examples of inner products considered in page no. 2, all satisfy

$$\langle x, x \rangle = \|x\|_2^2, \text{ where}$$

$$\|x\|_2 = \left( \sum_{j=1}^{\infty} |x(j)|^2 \right)^{1/2} \text{ or } \left( \int_E |x|^2 dm \right)^{1/2}.$$

In view of this, we explore the possibilities of constructing inner product from a norm and of obtaining a norm out of an inner product satisfying

$$\langle x, x \rangle = \|x\|^2 \quad \forall x \in X.$$

If  $\| \cdot \|$  is a norm on a linear space  $X$ , and we let  $d(x, y) = \|x-y\|$ , then  $d$  is a metric space on  $X$  and we can talk about the continuity of functions on  $X$ .

The above questions are answered by

Jordan-Von Neumann, 1935.

Theorem (Jordan-Von Neumann, 1935)

Let  $\|\cdot\|$  be a norm on a linear space  $X$ . Then there exists an inner product  $\langle \cdot, \cdot \rangle$  on  $X$  such that

$$\langle x, x \rangle = \|x\|^2 \text{ for all } x \in X$$

iff the norm satisfies the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad x, y \in X.$$

(The sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the ~~other~~ sides of the parallelogram).

In this case, such an inner product is unique and is given by

$$\langle x, y \rangle = \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2 + \|x+iy\|^2 - \|x-iy\|^2).$$

Proof. Let  $\langle \cdot, \cdot \rangle$  be an inner product on  $X$  such that  $\langle x, x \rangle = \|x\|^2$  for all  $x \in X$ . Then

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y \rangle + \langle x-y, x-y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &\quad + \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\ &= 2 \langle x, x \rangle + 2 \langle y, y \rangle \\ &= 2(\|x\|^2 + \|y\|^2), \end{aligned}$$

for all  $x, y \in X$ . Moreover, the polarization identity shows that

$$\begin{aligned} \langle x, y \rangle &= \frac{1}{4} [\langle x+y, x+y \rangle - \langle x-y, x-y \rangle + i \langle x+iy, x+iy \rangle \\ &\quad - i \langle x-iy, x-iy \rangle] \\ &= \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2 + i \|x+iy\|^2 - i \|x-iy\|^2) \end{aligned}$$

so that the above inner product is uniquely determined by  $\|\cdot\|$ .

Now, we prove the converse part of the theorem.