

Corollary: Let  $X$  be a Banach space under two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$ . If there exists  $\alpha > 0$  such that

$$\|x\|_1 \leq \alpha \|x\|_2 \quad \forall x \in X,$$

then there exists  $\beta > 0$  such that

$$\|x\|_2 \leq \beta \|x\|_1 \quad \forall x \in X.$$

In other words, any two comparable complete norms on a linear space  $X$  are equivalent.

### Open Mapping Theorem

We have seen in \*Metric Space section that if  $f$  is a continuous, one to one map from a metric space  $X$  onto a metric space  $Y$ , then it may not map from open subsets of  $X$  onto open subsets of  $Y$ .

In the next theorem, we shall prove that this is impossible if  $X$  and  $Y$  are Banach spaces and  $F$  is linear.

Theorem: The Open Mapping Theorem (Banach, 1932).

Let  $X$  and  $Y$  be Banach spaces and  $F: X \rightarrow Y$  linear. Then  $F$  is continuous and maps open subsets of  $X$  onto open subsets of  $Y$  iff the graph of  $f$  is closed in  $X \times Y$  and  $f$  is onto.

Proof: Let  $f$  be continuous and map open subsets of  $X$  onto open subsets of  $Y$ . Then, obviously,  $f$  has a closed graph in  $X \times Y$ . Also,  $f$  is onto, which can be shown as follows: there exists  $\delta > 0$  such that  $U_Y(0, \delta) \subset f(U_X(0, 1))$ . If  $y \in Y$  then  $\frac{\delta}{2} \|y\| \in U_Y(0, \delta)$  which is also in the range of  $f$ . Since  $f$  is linear, then  $y$  is also in the range of  $f$ .

Conversely, assume that  $f$  has a closed graph in  $X \times Y$  and is onto. Then by the Closed Graph Theorem,

$f$  is continuous. If, in addition,  $f$  is one to one,  
then by the previous Corollary (first corollary to  
Closed Graph Theorem),  $f$  is a linear homeomorphism  
so that it maps open subsets of  $X$  onto open  
subsets of  $Y$ . If  $f$  is not one-to-one, then we  
proceed as follows: Let  $Z = Z(f)$ . Since  $f$  is continuous,  
 $Z$  is closed in  $X$ . Hence the quotient space  $X/Z$  is  
a Banach space under the quotient norm  $\|\cdot\|_1$ . Now,  
let  $\tilde{f}: X/Z \rightarrow Y$  be given by

$$\tilde{f}(x+Z) = f(x).$$

Then  $\tilde{f}$  is well defined, linear, onto and also one-to  
one. Moreover,  $\tilde{f}$  has a closed graph in  $X/Z \times Y$ .  
To show this, let  $x_n + Z \rightarrow x + Z$  in  $X/Z$  and  $\tilde{f}(x_n + Z) \rightarrow y$   
in  $Y$ . Then there exists a sequence  $\{z_n\} \subset Z$  such that  
 $x_n + z_n \rightarrow x$  in  $X$ . Since  $f$  is continuous,

$$\tilde{f}(x_n + Z) = f(x_n + z_n) \rightarrow f(x),$$

which shows that  $y = \tilde{f}(x+Z)$ . Now, since  $\tilde{f}$  is one-to-one,  
then  $\tilde{f}$  maps open sets in  $X/Z$  onto open sets in  $Y$ .  
If  $E$  is an open subset of  $X$ , then

$$\tilde{E} = \{x+Z; x \in E\}$$

is  $\tilde{E}$  an open subset of  $X/Z$ , which can be proved as follows:

Let  $x_0 + Z \in \tilde{E}$  for  $x_0 \in E$ . Since  $E$  is open, there exists  
 $r > 0$  such that  $x \in E$  whenever  $\|x - x_0\| < r$ . ~~such that~~  
We claim that  $x + Z \in \tilde{E}$  whenever

$$\begin{aligned}\|(x+Z) - (x_0+Z)\|_1 &= \|(x-x_0) + Z\|_1 \\ &= \inf \{ \|x-x_0+z\|; z \in Z \} < r,\end{aligned}$$

then there exists  $z_0 \in Z$  such that  $\|x-x_0+z_0\| < r$ . Hence  
 $x+z_0 \in E$  which shows that

$$x_0 + z = x_0 + \tilde{z} \in \tilde{E}.$$

Then,  $\tilde{E}$  is open in  $X/\mathbb{Z}$ , and hence  $f(E) = \tilde{f}(\tilde{E})$  is open in  $Y$ . This proves the theorem.

Corollary: Let  $X$  be a Banach space and  $F \in BL(X)$ .

Then  $f^{-1}$  exists and belongs to  $BL(X)$  iff  $f$  is one to one and onto.