

Corollary: Let X be a Banach space, \mathbb{Y} a nls. and $\{f_n\}$ be a sequence in $BL(X, \mathbb{Y})$ such that $\{f_n(x)\}$ converges in \mathbb{Y} for every $x \in X$. For $x \in X$, let $f(x) = \lim f_n(x)$. Then we have the following properties:

(a) (Banach-Steinhaus Theorem, 1927).

$F \in BL(X, \mathbb{Y})$ and

$$\|F\| \leq \lim \limits_{n \rightarrow \infty} \|f_n\| \leq \sup \limits_{n=1,2,3,\dots} \{\|f_n\|\} < \infty,$$

(b) If E is any totally bounded subset of X , then $\{f_n(x)\}$ converges to $f(x)$ uniformly for $x \in E$.

Closed Graph Theorem

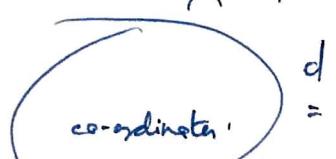
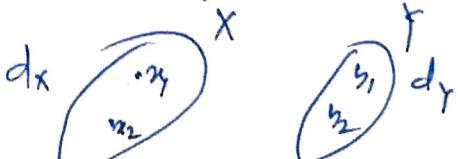
We have seen: if a linear map sends bounded sets in a nls. to bounded sets in another nls., then the linear map is continuous.

Now, we explore other criterions for the continuity of linear maps from a Banach space to a Banach space.

Definition

If f is a function from a set X to a set \mathbb{Y} , then the graph of f is the set $\{(x, f(x)) \in X \times \mathbb{Y} : x \in X\}$.

If X and \mathbb{Y} are metric spaces and we consider the metric $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$ on $X \times \mathbb{Y}$, then the graph of f would be closed in $X \times \mathbb{Y}$ iff $f(x) = y$ whenever $x_n \rightarrow x$ in X and $f(x_n) \rightarrow y$ in \mathbb{Y} . In particular, if f is a continuous function from X to \mathbb{Y} , then the graph of f is closed in $X \times \mathbb{Y}$. The converse is not true, for example let $X = \mathbb{Y} = \mathbb{R}$ and, $f(t) = y_t$ for $t \neq 0$ and $f(0) = 0$ shows. we show that such a situation does not arise for linear maps.



Lemma: Let X be a nls, $V = \{x \in X; \|x\| < 1\}$ and $\gamma > 1$. Let E be a subset of X such that for every $x \in V$ there exists $a \in E$ with $\|x - a\| < \gamma\delta$. Then for every $x \in V$ there exists a sequence $\{a_n\}$ in E such that

$$\left\| x - \sum_{j=0}^n \left(\frac{a_j}{\delta^j} \right) \right\| < \frac{1}{\delta^{n+1}}, \quad n = 0, 1, 2, \dots$$

Proof: Let $x \in V$. Then there exists $a_0 \in E$ such that $\|x - a_0\| < \gamma\delta$.

so, by the definition of V , $\underline{\sigma}(x - a_0) \in V$. Hence there exists $a_1 \in V$ such that

$$\|\underline{\sigma}(x - a_0) - a_1\| < \gamma\delta,$$

that is,

$$\|x - a_0 - a_1/\delta\| < \gamma\delta^2.$$

Proceeding similarly and having chosen a_0, a_1, \dots, a_{n-1} in E such that $\|x - a_0 - a_1/\delta - a_2/\delta^2 - \dots - a_{n-1}/\delta^{n-1}\| < \gamma\delta^n$,

$$\Rightarrow \underline{\sigma}(x - a_0 - a_1/\delta - a_2/\delta^2 - \dots - a_{n-1}/\delta^{n-1}) \in E$$

Hence there exists $a_n \in E$ such that $\|\underline{\sigma}(x - a_0 - a_1/\delta - a_2/\delta^2 - \dots - a_{n-1}/\delta^{n-1}) - a_n\| < \gamma\delta$ holds.

$$\Rightarrow \|x - a_0 - a_1/\delta - a_2/\delta^2 - \dots - a_{n-1}/\delta^{n-1} - a_n/\delta^n\| < \gamma\delta^{n+1}$$

Thus, we have established inductively the existence of a sequence $\{a_n\}$ in E with the stated property.

The closed Graph Theorem (Banach, 1932).

Let X and Y be Banach spaces and $F: X \rightarrow Y$ be a linear map which has a closed graph in $X \times Y$. Then $F \in BL(X, Y)$.

Proof: For each $\alpha > 0$, let $V_\alpha = \{x \in X; \|F(x)\| \leq \alpha\}$. If $V = \{x \in X; \|x\| < 1\}$, then we show that $V \subset V_\alpha$ for some $\alpha > 0$. Now,

$$X = \bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} \bar{V}_n$$

so that $\varphi = \bigcap_{n=1}^{\infty} (\bar{V}_n)^c$, where $(\bar{V}_n)^c$ denotes the

complement of \bar{V}_n . Since X is complete, then Baire's theorem shows that for some $n_0 \in \mathbb{N}$, $(\bar{V}_{n_0})^c$ is not dense in X ; i.e., there exists $x \in X$ and $\epsilon > 0$ such that $U(x, \epsilon) \subset \bar{V}_{n_0}$. Then $U(0, \epsilon) \subset \bar{V}_{2n_0}$ and $U \subset \bar{V}_{2n_0/\epsilon}$. Hence the set $V_{2n_0/\epsilon}$ has the property that for every $x \in U$, there exists $a \in V_{2n_0/\epsilon}$ such that $\|x - a\| < \gamma_0$. Let $a \in U$. Then by the previous lemma, there exists a sequence $\{a_n\}$ in $V_{2n_0/\epsilon}$ such that $\|x - a_n\| < \gamma_{2n_0+1}$, where $b_n = \sum_{j=0}^n a_j / 2^j$. Now, for $n = 0, 1, 2, \dots$,

$$\|F(b_n) - F(b_{n-1})\| = \|F(a_n/2^n)\| \leq 2^{n_0/\epsilon} \cdot 2^n = \epsilon,$$

Hence $\{F(b_n)\}$ is a Cauchy sequence in Y , and since Y is complete, it converges to some $y \in Y$. Since $b_n \rightarrow x$, $F(b_n) \rightarrow y$ and F is a closed graph, we see that $F(x) = y$. Thus,

$$\|f(x)\| = \lim \|F(b_n)\| \leq \sum_{j=0}^n \|F(a_j)\| / 2^j \leq \frac{4}{\epsilon} \cdot \frac{4^{n_0}}{\epsilon}.$$

Since $x \in U$ is arbitrary, then $U \subset V_{2n_0/\epsilon}$. Then, for $F \in U$, we have $F \in V_{2n_0/\epsilon}$ and hence bounded. Consequently, $F \in BL(X, Y)$. The theorem is proved.

Corollary: Let X and Y be Banach spaces and $F: X \rightarrow Y$ linear. If the graph of F is closed in $X \times Y$ and if F is one to one onto, then F is a linear homeomorphism from X onto Y .

Proof: