

Uniform Boundedness Principle

If \mathcal{F}_c be the set of continuous functions from a metric space X to a metric space Y , then uniform boundedness is a much stronger condition than its pointwise condition.

for example, let $X = [0, 1]$, $Y = \mathbb{R}$.

$\mathcal{F}_c = \{f_n, n=1, 2, 3, \dots\}$, where

$$f_n(t) = \begin{cases} n^2 t, & 0 \leq t \leq y_n \\ y_t, & y_n < t \leq 1. \end{cases}$$

$\rightarrow f_n(t)$ is continuous at $t = y_n$. (Verify).

$\rightarrow f_n(0) = 0$ and for $n \neq 0$, we have $\therefore n \leq \frac{1}{t}$

$$|f_n(t)| = |n^2 t| \leq \frac{1}{t^2} \cdot t = y_t$$

and $|f_n(t)| \leq \frac{1}{t}$ for $n > \frac{1}{t}$,

for $n = 1, 2, \dots$. Thus \mathcal{F} is pointwise bounded.

On the other hand, since

$$I = \sup \{ |f_n(t)| ; t \in X \} = \infty,$$

then I is not uniformly bounded.

clearly, Ascoli's theorem states that if X is compact and \mathcal{F}_c is equicontinuous at each point of X , then \mathcal{F}_c is uniformly bounded iff it is pointwise bounded.

- Since compactness and equicontinuity are hard to define in the settings of a nle X , we try to find another set of conditions for deducing uniform boundedness of a given family of functions/continuous maps on X .

Theorem L. (The Uniform Boundedness Principle, Banach, 1932)

Let X be a Banach space, Y a nls and F_c a family of bounded linear maps from X to Y . Then either there exists a dense subset D of X such that for every $x \in D$, the set

$$\{ \|f(x)\| : f \in F_c \}$$

is unbounded, or the norms of the maps in F_c are bounded.

Remark: The above theorem gives a condition for obtaining uniform boundedness from pointwise boundedness. We shall see this in the proof of the above theorem.

Proof of Theorem L:

$$\text{For } x \in X, \text{ let } s(x) = \sup \{ \|f(x)\| : f \in F_c \}.$$

Let \bar{U} denote a closed unit ball in X , and for

$n = 1, 2, 3, \dots$, let

$$E_n = \{ x \in \bar{U} : s(x) > n \}.$$

Then $E_n = \cup \{ x \in \bar{U} : \|f(x)\| > n \}$, where the union is taken over all $f \in F_c$. Since $f \in F_c$ is continuous (by the beginning theorem a continuity of nbs.), then $\{ x \in \bar{U} : \|f(x)\| > n \}$ is open in \bar{U} . Hence E_n is open in \bar{U} for all n .

Since \bar{U} is a closed subset of X , then \bar{U} is complete. If each E_n is dense in \bar{U} , then by Baire's theorem, the intersection $\bigcap_{n=1}^{\infty} E_n$ of dense open sets is dense in \bar{U} , ~~thus~~ and so -

$s(x) \rightarrow \infty$ for all $x \in \bigcap_{n=1}^{\infty} E_n$.

Since $s(kx) = |k| s(x)$, then $s(x) = \infty$ for all $x \in D = \{kx; 0 \neq k \in K, x \in \bigcap_{n=1}^{\infty} E_n\}$, which is dense in X .

On the other hand, if E_n is not dense in \bar{U} for some n_0 , then $\exists s \in \bar{U}$ and $0 < r < 1$ such that $\|f(b)\| \leq n_0$ for all $f \in F_c$ and all $b \in \bar{U}$ with $\|b-a\| < \epsilon$. Since \bar{U} is bounded, find $\epsilon' \in (0, r)$ such that $\bar{U} \subset U(a, \epsilon'/r)$ (Hint: take $r = \frac{\epsilon}{\|a\|+1}$).

Let $x \in \bar{U}$. If $b = rx + (1-r)a$, then $b \in \bar{U}$ and $\|b-a\| = r\|x-a\| < r \cdot \frac{\epsilon}{r} = \epsilon$. Hence, for all $f \in F_c$, $\|f(b)\| \leq n_0$ so that

$$\|f(x)\| = \|f(b) - (1-r)f(a)\|/r \leq \frac{n_0}{r}.$$

Thus, $s(x) \leq \frac{n_0}{r}$ for all $x \in \bar{U}$. Now, for any $f \in F_c$,

$$\|f\| = \sup \{ \|f(x)\| : x \in \bar{U} \} \leq \sup \{ s(x) : x \in \bar{U} \} \leq \frac{n_0}{r}.$$

Hence, in this case, the norms of the maps in F_c are bounded. The theorem is proved.

Geometrically: Either each $f \in F_c$ maps the closed unit-ball of a Banach space X into a fixed ball about 0 in Y , or else there exists $x \in X$ such that no ball in Y contains all $f(x), f \in F_c$.

