

# IMM6003 - Real Analysis and Topology

## Module IV-V

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# Real Numbers

In this Section, we shall study the following:

Field Structure and Order Structure of real numbers

Open and closed intervals

Bounded and unbounded sets-Supremum and Infimum

Order Completeness property

Archimedean Property of real Numbers

Open sets, Closed sets and Countable sets.

# Field Structure and Order Structure of real numbers

I hope you are well acquainted with Set, Subset, equality of sets, union and intersection of sets, and union and intersection of an arbitrary family of sets, Universal sets, functions, equivalent sets and composition of structure, that is, addition and multiplication.

# Field Structure and Order Structure of real numbers

## Field Structure:

A set  $S$  is said to be a field if two compositions of addition and multiplication defined in it be such that  $\forall a, b, c \in S$ , the following properties are satisfied:

**A.1-** Set  $S$  is closed under addition:  $a, b \in S \Rightarrow a + b \in S$ .

**A.2-** Set  $S$  is commutative under addition:  $a + b = b + a$ .

**A.3-** Set  $S$  is associative under addition:  $(a + b) + c = a + (b + c)$ .

**A.4-** Additive identity exists:  $\exists$  a number  $0 \in S$  such that  $a + 0 = a$ .

**A.5-** Additive inverse exists: to each element  $a \in S$ , there exists  $-a \in S$  such that  $a + (-a) = 0$ .

# Field Structure and Order Structure of real numbers

## Field Structure: Contd...

**M.1-** Set  $S$  is closed under multiplication:  $a, b \in S \Rightarrow ab \in S$ .

**M.2-** Set  $S$  is commutative under multiplication:  $ab = ba$ .

**M.3-** Set  $S$  is associative under multiplication:  $(ab)c = a(bc)$ .

**M.4-** Multiplicative identity exists:  $\exists$  a number  $1 \in S$  such that  $a.1 = a$ .

**M.5-** Multiplicative inverse exists: to each element  $a \in S$ , there exists  $a^{-1} \in S$  such that  $aa^{-1} = 1$ .

**AM-** Multiplication is distributive w.r.to addition, that is,  
 $a(b + c) = ab + ac$ .

# Field Structure and Order Structure of real numbers

## Order Structure:

A field  $S$  is an Ordered field if it satisfies the following properties:

**O.1-** Law of Trichotomy: For any two elements  $a, b \in S$  one and only one of the following holds:

$$a > b, a = b, a < b.$$

**O.2-** Transitivity:  $\forall a, b, c \in S,$

$$a > b \wedge b > c \Rightarrow a > c.$$

**O.3-** Compatibility of order relation with addition composition

:  $\forall a, b, c \in S,$

$$a > b \Rightarrow a + c > b + c.$$

**O.4-** Compatibility of order relation with multiplication composition

:  $\forall a, b, c \in S,$

$$a > b \wedge c > 0 \Rightarrow ac > bc.$$

# Field Structure and Order Structure of real numbers

## Order Structure: Examples:

$\mathbf{Q}, \mathbf{R}$ - Order fields

$\mathbf{N}, \mathbf{I}$ - not fields.

→ Can you give reasons behind this? Simple answers. See the definition and find the property which is not applicable in  $\mathbf{I}$  and  $\mathbf{N}$ .

→ **Hint:** Can you prove **A.5** and **M.5**? Nope !!!! So  $\mathbf{N}, \mathbf{I}$  are not fields.

# Field Structure and Order Structure of real numbers

## Well Ordering Principle for $\mathbf{N}$ :

Let  $\mathbf{N}$  be the set of Natural numbers. Then the following five properties hold (Peano's postulate for  $\mathbf{N}$ ):

**P-1:**  $1 \in \mathbf{N}$ , that is,  $\mathbf{N}$  is a nonempty set and contains an element we designate as 1,

**P-2:** For each element  $n \in \mathbf{N}$ , there is a unique element  $n_0 \in \mathbf{N}$  called the successor of  $n$ ,

**P-3:** For each  $n \in \mathbf{N}$ ,  $n_0 \neq 1$ , that is 1 is not the successor of any element in  $\mathbf{N}$ ,

**P-4:** For each pair  $n, m \in \mathbf{N}$  with  $n \neq m$ ,  $n_0 \neq m_0$ , that is, distinct elements in  $\mathbf{N}$  have distinct successors.

**P-5:** If  $A \subseteq \mathbf{N}$ ,  $1 \in A$  and  $p \in A$  implies that  $p_0 \in A$ , then  $A = \mathbf{N}$ .



# Field Structure and Order Structure of real numbers

## Well Ordering Principle for $\mathbb{N}$ : Contd...

→ **P-5** is called the *Principle of Mathematical Induction (PMI)*. From this PMI, we have the following statement : *Every nonempty subset of natural numbers has a first element (Well Ordering Principle)*.

# Intervals-Open and Closed

## Intervals-Open and Closed

A subset  $A$  of  $\mathbf{R}$  is called an **interval** if  $A$  contains (i) at least two distinct elements and (ii) every element lies between any two members of  $A$ .

**Open Interval:** If  $a$  and  $b$  are two real numbers such that  $a < b$ , then the set

$$\{x : a < x < b\}$$

consisting of all real numbers between  $a$  and  $b$  (excluding  $a$  and  $b$ ) is an open interval and is denoted by  $(a, b)$ .

**Closed Interval:** The set

$$\{x : a \leq x \leq b\}$$

consisting of  $a, b$  and all real numbers lying between  $a$  and  $b$  is called a closed Interval and is denoted by  $[a, b]$ .

→ In a similar way, one can define semi-open and semi-closed Intervals.

# Bounded and Unbounded Sets: Supremum and Infimum

## Supremum and Infimum

→ A subset  $S$  of Real numbers is said to be **bounded above** if  $\exists$  a real number  $K$  such that every member of  $S$  is less than or equal to  $K$ , that is,  $x \leq K, \forall x \in S$ .

→ The number  $K$  is called an **upper bound** of  $S$ .

→ If no such number  $K$  exists, the set is said to be **unbounded above** or **not bounded above**.

→ A subset  $S$  of Real numbers is said to be **bounded below** if  $\exists$  a real number  $k$  such that every member of  $S$  is greater than or equal to  $k$ , that is,  $x \geq k, \forall x \in S$ .

→ The number  $k$  is called an **lower bound** of  $S$ .

→ If no such number  $k$  exists, the set is said to be **unbounded below** or **not bounded below**.

→ A set is said to be bounded if it is bounded above and bounded below.

# Bounded and Unbounded Sets: Supremum and Infimum

## Supremum and Infimum

→ It may be observed that if a set has one upper bound, it has an infinite number of upper bounds. For, if  $K$  is an upper bound of a set  $S$  then every member greater than or equal to  $K$  is also an upper bound of  $S$ .

→ Thus every set  $S$  bounded above determines an infinite set — the set of its upper bounds. This set of upper bounds is bounded below in as much as every member of  $S$  is a lower bound thereof.

→ Similarly, a set  $S$  bounded below determines an infinite set of its lower bounds, which is bounded above by the members of  $S$ .

→ A member  $G$  of a set  $S$  is called the **greatest** member of  $S$  if every member of  $S$  is less than or equal to  $G$ , that is,  $G \in S$  and  $x \leq G \forall x \in S$ .

→ Similarly, a member  $g$  of the set is its **lowest** (or the least) member if every member of the set is greater than or equal to  $g$ .

→ Thus, a finite has a greatest and a least member.

# Bounded and Unbounded Sets: Supremum and Infimum

## Supremum and Infimum

→ If the set of all upper bounds of a set  $S$  has the smallest member, say  $M$ , then  $M$  is called the **least upper bound** (*l.u.b*) or the **supremum** of  $S$ .

→ Clearly, the supremum of a set  $S$  may or may not exist and in case it exists, it may or may not belong to the set  $S$ . The fact that supremum  $M$  is the smallest of all the upper bounds of  $S$  may be described by the following two properties:

- (i)  $M$  is the upper bound of  $S$ , that is,  $x \leq M \forall x \in S$ ,
- (ii) No member less than  $M$  can be an upper bound of  $S$ , that is, for any positive number  $\epsilon$ , however small,  $\exists$  a number  $y \in S$  such that  $y > M - \epsilon$ .

# Bounded and Unbounded Sets: Supremum and Infimum

## Properties of Supremum and Infimum

→ **Theorem** *A set cannot have more than one supremum.*

→ **Proof:** If possible, suppose that  $M$  and  $M'$  be two suprema of a set  $S$ , so that both  $M$  and  $M'$  are upper bounds of  $S$ . Since  $M$  is a l.u.b and  $M'$  is an upper bound, then  $M \leq M'$ . On the other hand,  $M$  is the upper bound and  $M'$  is the least upper bound, then  $M' \leq M$ . Consequently,  $M = M'$ . The theorem is proved.

→ In a similar way, we can define Infimum and their properties.....

→ I leave them to you as a home task.

→ Any Question(s) ?

→ Dont worry !!! I will be with you... to answer.

→ So want task(s) to do at home ? Here we go...

# Bounded and Unbounded Sets: Supremum and Infimum

## Supremum and Infimum: Home work: Compulsory

→ Prove that the greatest member of a set  $S$ , if it exists, is the supremum (*l.u.b.*) of the set.

→ Next Lecture: Order-Completeness in  $\mathbf{R}$  and Archimedean property of Real numbers. You will get soon.

# Completeness in the Set of Real numbers

## Order-Completeness in $\mathbf{R}$ :

*OC : Every non-empty set of real numbers which is bounded above has the supremum (or the least upper bound, that is, l.u.b) in  $\mathbf{R}$ .*

→ Note:  $\mathbf{Q}$  set fails here. This the reason why we go through Order Completeness property in  $\mathbf{R}$ .

→ In other words, the set of all upper bounds of a non-empty set of real numbers bounded above has the smallest number.

→ Lets make small modifications/additions or small manipulations on the above definition.

→ If  $S$  is a set of real numbers which is bounded above, then by considering the set  $T = \{x : -x \in S\}$ , we may state the completeness property in the following alternative form:

*(OCA) : Every non-empty set of real numbers which is bounded below has the infimum (g.l.b). Or, equivalently, the set of lower bounds of a nonempty set of real numbers bounded below has the greatest member.*



# Completeness in the Set of Real numbers

**Theorem-1. The set of rational numbers is not order-complete.**

*Proof* : It is sufficient to show that there exists a non-empty set  $S$  of rational (a subset of  $\mathbf{Q}$ ) which is bounded above doesnot have a supremum in  $\mathbf{Q}$ , that is, no rational number exists which can be the supremum of  $S$ .

Let  $S$  be the set (a subset of  $\mathbf{Q}$ ) of all positive rational numbers whose square is less than 2, that is,

$$S = \{x : x \in \mathbf{Q}, x > 0, x^2 < 2\}.$$

The set  $S$  is nonempty, because  $1 \in S$ . Clearly, 2 us an upper bound of  $S$ . Therefore  $S$  is bounded above. Thus,  $S$  is a set of rational numbers, bounded above. Let, if possible, the rational number  $K$  be its least upper bound. Clearly  $K$  is positive. Since the Law of Trichotomy holds good for  $\mathbf{Q}$ , then only one of the following hods:

$$K^2 < 2, K^2 = 2, K^2 > 2.$$

# Completeness in the Set of Real numbers

**Proof Contd...** We consider all the cases and let's see what we get.

(i) Let  $K^2 < 2$ . Consider the positive rational number

$$y = \frac{4 + 3K}{3 + 2K}.$$

Then

$$K - y = \frac{2(K^2 - 2)}{3 + 2K} < 0 \Rightarrow y > K.$$

Also,

$$2 - y^2 = \frac{2 - K^2}{(3 + 2K)^2} > 0 \Rightarrow y^2 < 2 \Rightarrow y \in S.$$

Thus the number  $y$  of  $S$  is greater than  $K$ , so that  $K$  cannot be an upper bound of  $S$ , a contradiction.

(ii) Let  $K^2 = 2$ . Not possible because there exists no rational number whose square is equal to 2.

# Completeness in the Set of Real numbers

**Proof Contd...** We consider all the cases and let's see what we get.

(iii) Let  $K^2 > 2$ . Considering the positive rational number  $y$  as in the case (i), we can easily obtain that

$$y < K \quad y^2 > 2.$$

Hence, there exists an upper bound  $y$  of  $S$  smaller than the least upper bound  $K$ , which is a contradiction.

## Archimedean Property of Real numbers

**Theorem-2.** The real number field is Archimedean, i.e., if  $a$  and  $b$  are any two positive real numbers, then there exists a positive integer  $n$  such that  $na > b$ .

*Proof :* Let  $a$  and  $b$  be two positive real numbers. Let us suppose, if possible, that for all positive integers  $n \in I^+$ ,  $na < b$ .

Thus, the set  $S = \{na; n \in I^+\}$  is bounded above,  $b$  being an upper bound. By the completeness property of the ordered field of real numbers, the set  $S$  must have the supremum  $M$ . So

$$na \leq M, \forall n \in I^+.$$

$$\Rightarrow (n+1)a \leq M, \forall n \in I^+.$$

$$\Rightarrow na \leq M - a, \forall n \in I^+,$$

that is,  $M - a$  is an upper bound of  $S$ .

Thus a member,  $M - a$  less than the supremum  $M$  (*l.u.b*) is an upper bound of  $S$ , which is a contradiction. Hence our supposition is wrong.

The theorem is proved.

# Archimedean Property of Real numbers

**Corollary-1.** If  $a$  is a positive real number and  $b$  are any real number, then there exists a positive integer  $n$  such that  $na > b$ .

**Corollary-2.** For any positive real number  $b$ , there exists a positive integer  $n$  such that  $n > b$ .

→ The result follows by considering  $a = 1$ .

**Corollary-3.** For any  $\epsilon > 0$ , there exists a positive integer  $n$ , such that  $1/n < \epsilon$ .

→ The result follows by taking  $b = 1/\epsilon$  in *Corollary-2*.

# Archimedean Property of Real numbers

**Theorem-3. Every open interval  $(a, b)$  contains a rational number.**

**Proof.** We shall prove the theorem in three different cases. We shall see them one by one.

*Case-I:* If  $0 < a < b$ , then by Corollary-3, there is a  $m \in \mathbb{N}$  such that  $1/m < (b - a)$ . Let

$$A = \left\{ n \in \mathbb{N} : \frac{n}{m} > a \right\}.$$

By Archimedean Property,  $A \neq \emptyset$ . Now, by the well-Ordering principle for  $\mathbb{N}$ ,  $A$  has a first element, say,  $n_0$ , and so  $n_0 - 1 \notin A$ . Consequently,

$$\frac{n_0 - 1}{m} \leq a \Rightarrow \frac{n_0}{m} \leq a + \frac{1}{m} < a + (b - a).$$

$$\Rightarrow \frac{n_0}{m} < b. \text{ But } n_0 \in A. \text{ Hence } \frac{n_0}{m} > a.$$

Hence there exists a rational number  $\frac{n_0}{m}$  in the open interval  $(a, b)$ .

# Archimedean Property of Real numbers

## Proof: Contd...

*Case-II:* If  $a \leq 0 < b$ , then again by Corollary-3, there is a  $n \in \mathbb{N}$  with  $1/n < b$ . Clearly  $1/n \in (a, b)$ .

*Case-III:* If  $a < b \leq 0$ , then  $0 \leq -b < -a$ . By the previous cases, there is a rational number  $q \in (-b, -a)$ , and so the rational number  $-q \in (a, b)$ . The theorem is proved.

# Open Sets, Closed Sets and Countable Sets:

**Neighbourhood of a Point:** A set  $N \subseteq R$  is called the neighbourhood of a point  $a$ , if there exists an open interval  $I$  containing  $a$  and contained in  $N$ , that is,

$$a \in I \subseteq N.$$

→ It follows from the definition that an open interval is a neighbourhood of each of its points.

→ For our convenient, we consider  $(a - \delta, a + \delta)$  to be a nbhd. of  $a$ , where  $\delta > 0$ .

**Deleted Neighbourhood:** The set  $\{x : 0 < |x - a| < \delta\}$ , that is, an open interval  $(a - \delta, a + \delta)$  from which the number  $a$  itself has been *excluded or deleted* is called a *deleted nbhd.*



# Open Sets, Closed Sets and Countable Sets:

## Neighbourhood of a Point: Some Examples.

- The set  $R$  of real numbers is the nbhd. of each of its points.
- The set  $Q$  of rational is not a nbhd of any of its points.
- The open interval  $(a, b)$  is a nbhd. of each of its points.
- The closed interval  $[a, b]$  is the nbhd. of each point of  $(a, b)$  but is not a nbhd. of the end points  $a$  and  $b$ .
- The null set  $\Phi$  is a nbhd. of each of its points in the sense that there are no points in  $\Phi$  of which it is not a nbhd..

# Open Sets, Closed Sets and Countable Sets:

## Neighbourhood of a Point: Some Good Examples:

**Example-1** A non-empty finite set is not a nbhd. of any point.

*Proof:* A set can be a nbhd. of a point if it contains an open interval containing the point. Since an interval necessarily contains an infinite number of points, therefore, in order that a set be a nbhd. of a point it must (necessarily) contain an infinity of points. Thus a finite set cannot be a nbhd. of any point.

**Example-2** Superset of a nbhd. of a point  $x$  is also a nbhd. of the point  $x$ , that is, if  $N$  is a nbhd. of a point  $x$  and  $M \supseteq N$ , then  $M$  is also a nbhd. of  $x$ .

**Example-3** Union (finite or arbitrary) of nbhd. of a point  $x$  is also a nbhd. of  $x$ .

# Open Sets, Closed Sets and Countable Sets:

## Neighbourhood of a point: Some Good Examples: Contd...

**Example-4** If  $M$  and  $N$  are neighbourhoods of a point  $x$ , then show that  $M \cap N$  is also a nbhd. of  $x$ .

*Proof:* Since  $M$  and  $N$  are neighbourhoods of  $x$ , there exist open intervals enclosing the point  $x$  such that

$$x \in (x - \delta_1, x + \delta_1) \subseteq M \text{ and } x \in (x - \delta_2, x + \delta_2) \subseteq N.$$

Let  $\delta = \min\{\delta_1, \delta_2\}$ . Then

$$(x - \delta, x + \delta) \subseteq (x - \delta_1, x + \delta_1) \subseteq M$$

and

$$(x - \delta, x + \delta) \subseteq (x - \delta_2, x + \delta_2) \subseteq N.$$

Thus,  $(x - \delta, x + \delta) \subseteq M \cap N$ , that is,  $M \cap N$  is a nbhd. of  $x$ . (proved).

# Open Sets, Closed Sets and Countable Sets:

## Interior Points of a Set:

A point  $x$  is an interior point of a set  $S$  if  $S$  is a nbhd. of  $x$ . In other words,  $x$  is an interior point of  $S$ , if there exists an open interval  $(a, b)$  containing  $x$  and contained in  $S$ , that is,  $x \in (a, b) \subseteq S$ .

→ Thus a set is a nbhd. of each of its interior points.

**Interior of a Set:** The set of all interior points of a set is called a **interior** of the set. The interior of a set  $S$  is, generally, denoted by  $S^i$  or  $\text{int } S$ .

→ **Homework:** Show that the interior of the set **N** or **I** or **Q** is the null set, but interior of **R** is **R**.

→ **Homework:** Show that the interior of a set  $S$  is a subset of  $S$ , that is,  $S^i \subseteq S$ .

# Open Sets, Closed Sets and Countable Sets:

## Open Set

A set  $S$  is said to be open if it is a nbhd. of each of its points, that is, for each  $x \in S$ , there exists an open interval  $I_x$  such that  $x \in I_x \subseteq S$ .

→ Thus *every point of an open set is an interior point*, so that for an open set  $S$ ,  $S^i = S$ .

→ Thus,  $S$  is open  $\iff S^i = S$ .

→ Of course the set is **not open** if it is not a nbhd. of at least one of its points or that there is at least one point of the set which is not an interior point.

# Open Sets, Closed Sets and Countable Sets:

## Open Set: Some Illustrations:

- The set  $\mathbf{R}$  of real numbers is an open set.
- The set  $\mathbf{Q}$  of rationals is not an open set.
- The closed interval  $[a, b]$ , is not open for it is not a nbhd. of the end points  $a$  and  $b$ .
- The null set  $\Phi$  is open, for there is no point  $\Phi$  of which it is not a nbhd.
- A non-empty finite set is not open.
- The set  $\{\frac{1}{n} : n \in \mathbf{N}\}$  is not open.

**Homework** Give an example of an open interval which is not an interval.

# Open Sets, Closed Sets and Countable Sets:

## Open Set: Some Examples.

**Example-5** Show that every open interval is an open set. Or, every open interval is a nbhd. of each of its points.

**Proof:** Let  $x$  be any point of the given open interval  $(a, b)$ , so that  $a < x < b$ . Let  $c$  and  $d$  be two numbers such that

$$a < c < x \text{ and } x < d < b$$

so that we have

$$a < c < x < d < b \Rightarrow x \in (c, d) \subset (a, b).$$

Thus the given interval  $(a, b)$  contains an open interval containing the point  $x$ , and is therefore a nbhd. of  $x$ .

**Note:** *Open interval is a nbhd. of each of its points and is therefore an open set.*

# Open Sets, Closed Sets and Countable Sets:

## Open Set: Some Examples. Contd...

**Example-6** Show that every open set is a union of open intervals.

**Proof:** Let  $S$  be an open set and  $x_\lambda$  is a point of  $S$ . Since  $S$  is open, then there exists an open interval  $I_{x_\lambda}$  for each of its points  $x_\lambda$  such that  $x_\lambda \in I_{x_\lambda} \subseteq S$  for all  $x_\lambda \in S$ .

Again the set  $S$  can be thought of as the union of singleton sets like  $\{x_\lambda\}$ , that is,  $S = \cup_{\lambda \in \Lambda} \{x_\lambda\}$ , where  $\Lambda$  is the index set. Thus

$$S = \cup_{\lambda \in \Lambda} \{x_\lambda\} \subseteq \cup_{\lambda \in \Lambda} I_{x_\lambda} \subseteq S.$$

$$\Rightarrow S = \cup_{\lambda \in \Lambda} I_{x_\lambda}. (proved)$$



# Open Sets, Closed Sets and Countable Sets:

## Theorems on Open Sets:

**Theorem-1:** *The interior of a set is an open set.*

**Proof:** Let  $S$  be a given set, and  $S^i$  its interior. If  $S^i = \phi$ , then  $S^i$  is open. When  $S^i \neq \phi$ , let  $x$  be any point of  $S$ ,  $\exists$  an open interval  $I_x$  such that  $x \in I_x \subseteq S$ . But  $I_x$ , being an open interval, is a nbhd of each of its points. Thus every point of  $I_x$  is an interior point of  $S$ , and  $I_x \subseteq S^i$ . Hence  $I_x \subset S^i$ . This in turn implies that  $x \in I_x \subseteq S^i$  and hence any point  $x$  of  $S^i$  is an interior point of  $S^i$ . Thus  $S^i$  is an open set. The theorem is proved.

**Corollary:** The interior of a set  $S$  is an open subset of  $S$ .

# Open Sets, Closed Sets and Countable Sets:

## Theorems on Open Sets:

**Theorem-2:** *The interior of a set  $S$  is the largest open subset of  $S$ .*

*or*

The interior of a set  $S$  contains every open set of  $S$ .

**Corollary:** Interior of a set  $S$  is union of all open subsets of  $S$ .

**Theorem-3:** *The union of an arbitrary family of open sets is open.*

**Theorem-4:** *The intersection of any finite number of open sets is open.*

# Open Sets, Closed Sets and Countable Sets:

## Limit Points of a Set:

**Definition-1:** A number  $\xi$  is a limit point of a set  $S(\subset \mathbf{R})$  if every nbhd. of  $\xi$  contains an infinite number of members of  $S$ .

→ Thus,  $\xi$  is a limit point of a set  $S$  if for any nbhd.  $N$  of  $\xi$ ,  $N \cap S$  is an infinite set.

→ A limit point is also called a cluster point or an accumulation point.

→ A limit point may or may not be a member of the set. Further, it is clear from the definition that a finite set cannot have a limit point. Also, it is not necessary that an infinite set must possess a limit point. *In fact*, a set may have no limit point, a unique limit point, a finite or an infinite number of limit points. A sufficient condition for the existence of a limit point is provided by the *Bolzano-Weirstrass theorem* which we shall discuss later. We have another definition of a limit point, which is equivalent to *Definition-1*, we shall see this later.

# Open Sets, Closed Sets and Countable Sets:

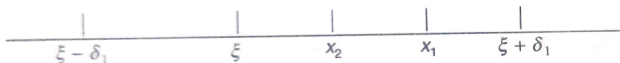
**Limit Points of a Set: Contd...** **Definition-2:** *A real number  $\xi$  is a limit point of a set  $S(\subset \mathbf{R})$  if every nbhd. of  $\xi$  contains at least one member of  $S$  other than  $\xi$ .*

→ The essential idea here is that the point of  $S$  different from  $\xi$  get arbitrary close" to  $\xi$ .

→ Clearly, Definition-1 implies Definition-2. We shall prove the converse.

# Open Sets, Closed Sets and Countable Sets:

## Definition-2 $\Rightarrow$ Definition-1



Let  $\xi$  be a point of the set  $S(\subset \mathbf{R})$  such that every nbhd. of  $\xi$  contains at least one point of  $S$  other than  $\xi$ . Let  $(\xi - \delta_1, \xi + \delta_1)$  be one such nbhd. of  $\xi$  which contains at least one point  $x_1 \neq \xi$  of  $S$ .

Let  $|x_1 - \xi| = \delta_2 < \delta_1$ . Now we consider the nbhd.  $(\xi - \delta_2, \xi + \delta_2)$  of  $\xi$  which by Definition-2 has a limit point, say  $x_2$  of  $S$  other than  $\xi$ .

By repeating the argument with the nbhd.  $(\xi - \delta_3, \xi + \delta_3)$  of  $\xi$  with  $|x_2 - \xi|$  and so on, it follows that the nbhd.  $(\xi - \delta_i, \xi + \delta_i)$  of  $\xi$  contains an infinity of members of  $S$ .

Thus Definition-2  $\Rightarrow$  Definition-1.

# Open Sets, Closed Sets and Countable Sets:

## Limit Point-Remarks

→ Note that a point  $\xi$  is not a limit point of a set  $S$  if  $\exists$  even one nbhd. of  $\xi$  not containing any point of  $S$  other than  $\xi$ .

**Derived Set:** The set of all limit points of a set  $S$  is called a *derived set* if  $S$  and is denoted by  $S'$ .

# Open Sets, Closed Sets and Countable Sets:

## Limit Point: Illustrations

→ The set  $\mathbf{I}$  has no limit point, for a nbhd.  $(m - 1/2, m + 1/2)$  of  $m \in \mathbf{I}$ , contains no point of  $\mathbf{I}$  other than  $m$ . Thus the derived set of  $\mathbf{I}$  is the null set  $\phi$ .

→ Every point of  $\mathbf{R}$  is a limit point, for, every nbhd. of any of its points contains an infinity of members of  $\mathbf{R}$ . Therefore  $\mathbf{R} = \mathbf{R}'$ .

→ Every point of the set  $\mathbf{Q}$  of rational is a limit point.

→ Homework: Prove that  $\mathbf{Q}' = \mathbf{R}$ .

→ The set  $\{\frac{1}{n} : n \in \mathbf{N}\}$  has only one limit point 0, which is not a member of the set.

→ Every point of the closed interval  $[a, b]$  is its limit point, and a point not belonging to the interval is not a limit point. Thus the derived set  $[a, b]' = [a, b]$ .

→ Every point of the open interval  $(a, b)$  is its limit point. The end points  $a$  and  $b$  which are not members of  $(a, b)$  are also its limit points. Thus  $(a, b)' = (a, b)$ .

# Open Sets, Closed Sets and Countable Sets:

**Bolzano-Weirstrass Theorem (for sets):** *Every infinite bounded set has a limit point.*

**Proof:** Let  $S$  be any bounded set and  $m, M$  its infimum and supremum respectively. Let  $P$  be the set of real numbers defined as follows:

$x \in P$  **iff it exceeds at the most a finite number of members of  $S$ .**

The set  $P$  is non-empty, for  $m \in P$ . Also  $M$  is an upper bound of  $P$ , for no number greater than or equal to  $M$  can belong to  $P$ . Thus, the set  $P$  is non-empty and is bounded above. Therefore, by order-completeness property,  $P$  has the supremum, say  $\xi$ . We shall now show that  $\xi$  is a limit point of  $S$ .

Consider the nbhd.  $(\xi - \epsilon, \xi + \epsilon)$  of  $\xi$ , where  $\epsilon > 0$ .

Since  $\xi$  is the supremum of  $P$ ,  $\exists$  at least one member, say  $\eta$  of  $P$  such that  $\eta > \xi - \epsilon$ . Now  $\eta$  belongs to  $P$ , therefore it exceeds at the most a finite number of members of  $S$ , and consequently  $\xi - \epsilon (< \eta)$  can exceed at the most a finite number of members of  $S$ .



# Open Sets, Closed Sets and Countable Sets:

## Bolzano-Weirstrass Theorem: Proof Contd...

Again as  $\xi$  is the supremum of  $P$ ,  $\xi + \epsilon$  cannot belong to  $P$ , and consequently  $\xi + \epsilon$  must exceed an infinite number of members of  $S$ .

Now  $\xi - \epsilon$  exceeds at the most of finite number of members of  $S$  and  $\xi + \epsilon$  exceeds infinitely many members of  $S$ .

$(\xi - \epsilon, \xi + \epsilon)$  contains an infinite number of members of  $S$ .  
Consequently,  $\xi$  is a limit point of  $S$ . The theorem is proved.

# Open Sets, Closed Sets and Countable Sets:

## Closed Sets: Closure of a Set

**Definition:** A real number  $\xi$  is said to be an **Adherent point** of a set  $S(\subseteq \mathbf{R})$  if every nbhd. of  $\xi$  contains at least one point of  $S$ .

→ Evidently, an adherent point may or may not belong to the set and it may or may not be a limit point of the set.

→ It follows from the definition that a number  $\xi \in S$  is automatically an adherent point of the set, for, every nbhd. of a member of a set contains at least one member of the set, namely the member itself.

→ Further, a number  $\xi \notin S$  is an adherent point of  $S$  only if  $\xi$  is a limit point of  $S$ , for, every nbhd. of  $\xi$  there contains at least one point of  $S$  which is other than  $\xi$ .

→ The set of adherent points of  $S$  consists of  $S$  and the derived set  $S'$ .

→ The set of all adherent points of  $S$  is called the closure of  $S$ , denoted by  $\overline{S}$ , and is such that  $\overline{S} = S \cup S'$ .

# Open Sets, Closed Sets and Countable Sets:

## Closed Sets

**Definition:** A set  $S$  is said to be **closed** if each of its limit points is a member of the set.

**Definition:** In Other words, a set  $S$  is said to be closed if no limit point of  $S$  exists which is not contained in  $S$ . In rough words, a set is closed if its points do not get arbitrarily close to any point outside of it.

→ Thus a set  $S$  is *closed* iff

$$s' \subseteq S \text{ or } \overline{S} = S.$$

**Remark:** The concept of closed and open sets are neither mutually exclusive nor exhanstive. The word ***not closed*** should not be considered equivalent to open set. Sets exist (i) both open and closed, (ii) neither open nor closed. The set consisting of points of  $(a, b)$  neither open nor closed.

# Open Sets, Closed Sets and Countable Sets:

## Closed Sets: Illustrations:

- (i)  $[a, b]$  is a set which is closed but not open.
- (ii) The set  $[0, 1] \cup [2, 3]$ , which is not an interval, is closed.
- (iii) The null set  $\Phi$  is closed because there exists no limit points of  $\Phi$  which is not contained in  $\Phi$ . As was shown earlier,  $\Phi$  is also open.
- (iv) The set  $\mathbf{R}$  of real numbers is open as well as closed.
- (v) The set  $\mathbf{Q}$  is not closed because  $\mathbf{Q}' = \mathbf{R} \not\subset \mathbf{Q}$ . Also, it is not open.
- (vi)  $\left\{\frac{1}{n}; n \in \mathbf{N}\right\}$  is not closed, because it has one limit point, 0, which is not a member of the set. Also it is not open.
- (vii) Every finite set  $A$  is a closed set because, its derived set  $A' = \Phi \subset A$ .
- (viii) A set  $S$  which has no limit point coincides with its closure, because  $A' = \Phi$  and  $\overline{A} = A \cup A' = A$ .

# Open Sets, Closed Sets and Countable Sets:

## Closed Sets: Typical Examples

**Example-1** Show that the set  $S = \{x : 0 < x < 1, x \in \mathbf{R}\}$  is open but not closed.

In other words, the set  $S$  is the open interval  $(0, 1)$ . So, it contains a nbhd. of each of its points. Hence it is an open set.

Again, every point of  $S$  is a limit point. The end points 0 and 1 which are not members of the set are also limit points. Thus, the set  $S$  is not closed.

# Open Sets, Closed Sets and Countable Sets:

## Closed Sets: Typical Examples: Home Work/Assignments

**Example-2** Show that the set  $S = \{1, -1, 1/2, -1/2, 1/3, -1/3, \dots\}$  is neither open nor closed.

**Example-3** Show that the set  $S = \{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \dots\}$  is closed but not open.

# Open Sets, Closed Sets and Countable Sets:

## Dense Sets

A subset  $A$  of the set of reals  $\mathbf{R}$  is said to be *dense* (or dense in  $\mathbf{R}$  or everywhere dense) if every point of  $\mathbf{R}$  is a point of  $A$  or a limit point of  $A$  or equivalently if the closure of  $A$  is  $\mathbf{R}$ .

A set  $A$  is said to be dense in itself if every point of  $A$  is a limit point of  $A$ , i.e., if  $A \subseteq A'$ . A set which is dense in itself has no isolated points.

A set  $A$  is said to be nowhere dense (nondense) relative to  $\mathbf{R}$  if no neighbourhood in  $\mathbf{R}$  is contained in the closure of  $A$ . In other words if the complement of the closure of  $A$  is dense in  $\mathbf{R}$ .

It is clear that if  $A$  is an interval or contains an interval then  $A$  is not nowhere dense. Because there exists an interval  $I \in \mathbf{R}$  such that  $I \cap A \neq \Phi$ . But there are sets which contain no interval and which fail to be nowhere dense; for example the set of rationals  $\mathbf{Q}$  and the set of irrationals  $\mathbf{R-Q}$ .

A set is said to be perfect if it is identical with its derived set or equivalently a set which is closed and dense in itself

# Open Sets, Closed Sets and Countable Sets:

## Closed Sets: Some Important Theorems

**Theorem:** A set is closed *iff* its complement is open.

**Theorem:** The intersection of an arbitrary family of closed sets is closed.

**Theorem:** The union of two closed sets is a closed set.

**Theorem:** The derived set of a set is closed.

**Theorem:** A closed set either contains an open interval or else is nowhere dense.

**Theorem:** The derived set of a bounded set is bounded.



# Open Sets, Closed Sets and Countable Sets:

## Countable and Uncountable Sets

**Definition:** An infinite set  $A$  is said to be countably infinite (or denumerable or enumerable) if it is equivalent to the set  $\mathbf{N}$  of natural numbers.

A set which is either empty or finite or countably infinite is called countable; otherwise it is uncountable.