

Existence of Limit Points

We note that members of a sequence forms a set (the range set / subset of the larger set). Thus, all theorems relating to bounds and limit points also works for sequences, with suitable modifications.

Bolzano-Weierstrass Theorem:

Every bounded sequence has a limit point.

Proof. Let $\{s_n\}$ be a bounded sequence and

$$S = \{s_n; n \in \mathbb{N}\}$$

be its range. Since the sequence is bounded, then its range set is also bounded.

Now, we have two possibilities:

(i) S is finite

(ii) S is infinite

We deal with ~~all~~ both the possibilities given above:

(i) If S is finite, then there exist at least one ~~non~~ member $\xi \in S$ such that $s_n = \xi$ for an infinite number of values of n . This means that every nbhd $(\xi - \epsilon, \xi + \epsilon)$ of ξ contains $s_n (= \xi)$ for an infinite number of values of n .

Thus, ξ is the limit point of $\{s_n\}$.

(ii) When S is infinite, since it is bounded, then by the Bolzano-Weierstrass Theorem for sets, it has a limit point; say ξ .

Again, since ξ is a limit point of S , then every nbhd. $(\xi - \epsilon, \xi + \epsilon)$ of ξ contains an infinite number of s , that is, $s_n \in (\xi - \epsilon, \xi + \epsilon)$ for infinite values of n . Thus, ξ is a limit point of the sequence.

Remark The converse of the above theorem is not necessarily true. For example, the sequence

$$\{1, 2, 1, 4, 1, 6, 1, 8, \dots\}$$

has a unique limit point 1 but not bounded above.

Theorem: The set of limit points of a bounded sequence has the greatest and the least member.

Remark: The greatest and the smallest of the limit points of a bounded sequence are respectively, called the upper limit and the lower limit.

Illustrations

(i) $\{s_n\}$, $s_n = (-1)^n$, n.e.n. bounded, $-1 \leq s_n \leq 1$ then,
 -1 and 1 are the limit points.
 Upper limit is 1 and lower limit is -1.

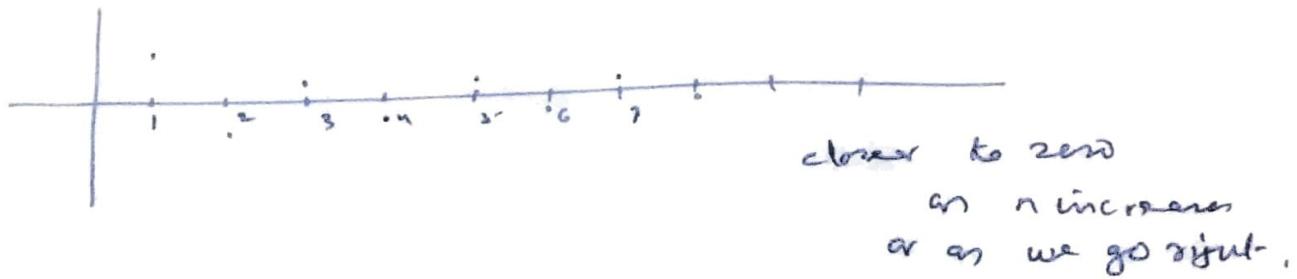
(ii) $\{s_n\}$, $s_n = 1 + (-1)^n$, n.e.n. bounded, $0 \leq s_n \leq 2$, then,
 0 and 2 are the limit points.
 Upper limit = 2, Lower limit = 0.

(iii) $\{s_n\}$, $s_n = \frac{(-1)^{n+1}}{n}$, n.e.n. $\left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots\right\}$ bounded.

$$-\frac{1}{2} \leq s_n \leq 1 \quad \forall n \in \mathbb{N}.$$

Here 0 is the limit point, upper limit & lower limit coincide with 0, and so

$$\lim_{n \rightarrow \infty} s_n = 0.$$



(iv) $\{s_n\}$, $s_n = n^2$, $n \in \mathbb{N}$. $\{1, 4, 9, 16, 25, \dots\}$

lower bound is 1, upper bound? so not bounded above. So the sequence $\{s_n\}$ has no limit point.

Limit Inferior and Limit Superior of Sequences:

We have observed from the definition of limits of a sequence, that the limiting behaviour of any sequences $\{a_n\}$ of real numbers, depends only ~~also~~ on sets of the form $\{a_n; n > m\}$, that is, $\{a_m, a_{m+1}, a_{m+2}, \dots\}$. In view of this, we give the following definition:

Definition:

Let $\{a_n\}$ be a sequence of real numbers (not necessarily bounded). We define

$$\liminf_{n \rightarrow \infty} a_n = \sup_n \{a_n, a_{n+1}, a_{n+2}, \dots\}.$$

$$\limsup_{n \rightarrow \infty} a_n = \inf_n \sup \{a_n, a_{n+1}, a_{n+2}, \dots\}.$$

as the limit inferior and limit superior respectively of the sequence $\{a_n\}$.

We denote:

limit inferior

$$\lim_{n \rightarrow \infty} a_n \quad \text{or} \quad \underline{\lim} a_n$$

limit superior

$$\overline{\lim}_{n \rightarrow \infty} a_n \quad \text{or} \quad \overline{\lim} a_n.$$

If we use the following notations for the sequence $\{a_n\}$,
for each $n \in \mathbb{N}$,

$$\underline{A}_n = \inf \{a_n, a_{n+1}, a_{n+2}, \dots\}.$$

$$\bar{A}_n = \sup \{a_n, a_{n+1}, a_{n+2}, \dots\},$$

then we have

$$\liminf a_n = \sup_n \underline{A}_n$$

$$\limsup a_n = \inf_n \bar{A}_n.$$

Remark:

Since $\{a_{n+1}, a_{n+2}, \dots\} \subseteq \{a_n, a_{n+1}, a_{n+2}, \dots\}$,

then taking limit sup. & limit inf both sides, we get

$$\underbrace{\underline{A}_{n+1}}_{\downarrow} \geq \underline{A}_n \quad \text{and} \quad \overbrace{\bar{A}_{n+1}}^{\downarrow} \leq \bar{A}_n.$$

$$\underline{A}_1 \leq \underline{A}_2 \leq \underline{A}_3 \leq \dots \leq \underline{A}_n \leq \underline{A}_{n+1} \dots$$

sequence $\{\underline{A}_n\}$ monotonically
increasing.

$$\dots \leq \bar{A}_{n+1} \leq \bar{A}_n \leq \bar{A}_{n-1} \dots \leq \bar{A}_3 \leq \bar{A}_2 \leq \bar{A}_1$$

sequence $\{\bar{A}_n\}$ is
monotonically decreasing.

Theorem: If $\{a_n\}$ is any sequence, then

$$\inf a_n \leq \liminf a_n \leq \limsup a_n \leq \sup a_n.$$

Corollary: If a sequence $\{a_n\}$ is bounded, then their
limit inferior & limit superior are both finite,

that is, $-\infty < \liminf a_n \leq \limsup a_n < \infty$.

Theorem If $\{a_n\}$ is any sequence, then

$$\underline{\lim} (-a_n) = - \overline{\lim} a_n$$

$$\text{and } \overline{\lim} (-a_n) = - \underline{\lim} a_n.$$

Illustrations:

(i) $a_n = (-1)^n, n \in \mathbb{N}, \{(-1)^n\}$. Then

$$a_n = -1 \text{ and } \bar{a}_n = 1. \quad \forall n \in \mathbb{N}.$$

$$\underline{\lim} a_n = \sup \bar{a}_n = -1, \quad \overline{\lim} a_n = \inf \bar{a}_n = 1.$$

(ii) $\left\{ 1 + (-1)^n \right\}_{n=1}^{\infty}, a_n = 1 + (-1)^n, \forall n \in \mathbb{N}$. Then

$$\bar{a}_n = \inf \left\{ 1 + (-1)^n, 1 + (-1)^{n+1}, 1 + (-1)^{n+2}, \dots \right\} = 0.$$

$$\text{and } \underline{a}_n = \sup \left\{ 1 + (-1)^n, 1 + (-1)^{n+1}, 1 + (-1)^{n+2}, \dots \right\} = 2$$

for each $n \in \mathbb{N}$. Hence

$$\underline{\lim} a_n = 0 \text{ and } \overline{\lim} a_n = 2.$$

Homework (To do yourself. Try to do yourself. If fails, then have discussion among your friends. If fails, again, go here to solve your problem/doubts.)

(i) $\{n\}_{n=1}^{\infty}$, (ii) $\{(-1)^n\}_{n \in \mathbb{N}}$, (iii) $\left\{ \frac{(-1)^n}{n^2} \right\}_{n \in \mathbb{N}}$.

$$\underline{\lim} a_n = \infty$$

$$\underline{\lim} a_n = -\infty$$

$$\underline{\lim} a_n = 0.$$

$$\overline{\lim} a_n = \infty$$

$$\overline{\lim} a_n = \infty$$

$$\overline{\lim} a_n = 0.$$

(iv) $\left\{ (-1)^n \left(1 + \frac{1}{n} \right) \right\}_{n \in \mathbb{N}}$, (v) $\{n(1 + (-1)^n)\}_{n \in \mathbb{N}}$, (vi) $\left\{ \sin \frac{n\pi}{3} \right\}_{n \in \mathbb{N}}$.

$$\underline{\lim} a_n = -1$$

$$\underline{\lim} a_n = 0$$

$$\underline{\lim} a_n = -\frac{\sqrt{3}}{2}$$

$$\overline{\lim} a_n = 1$$

$$\overline{\lim} a_n = \infty$$

$$\overline{\lim} a_n = \frac{\sqrt{3}}{2}$$