

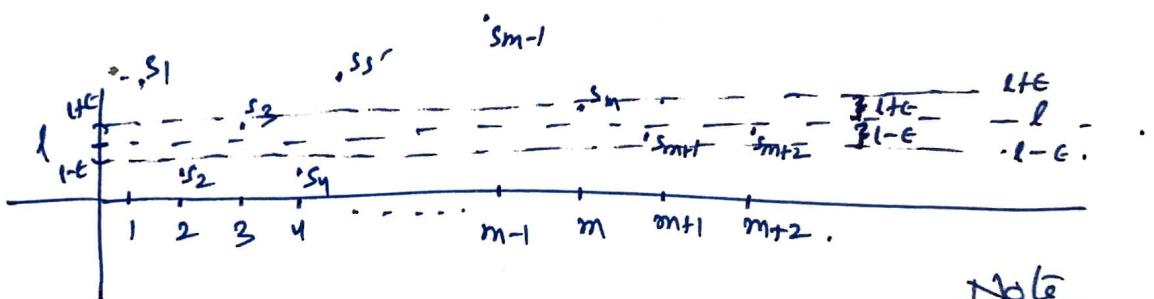
Theorem 1. Every convergent sequence is bounded.

Proof. Let a sequence  $\{s_n\}$  converge to the limit  $l$ .

Let  $\epsilon > 0$  be a given number for which there exists a positive integer  $m$  such that

$$|s_n - l| < \epsilon \text{ for } n \geq m.$$

or  $|l - \epsilon| < s_n < |l + \epsilon| \text{ for } n \geq m. \dots \dots \dots \quad (1)$



Let  $g = \min \{l - \epsilon, s_1, s_2, \dots, s_{m-1}\} \Rightarrow g \leq l - \epsilon$

and  $G = \max \{l + \epsilon, s_1, s_2, \dots, s_{m-1}\} \Rightarrow G \geq l + \epsilon$ .

Then we have (follows from (1)),

$$g \leq s_n \leq G \text{ for } n \geq 1.$$

Hence  $G$  is a bounded sequence. The theorem is proved.

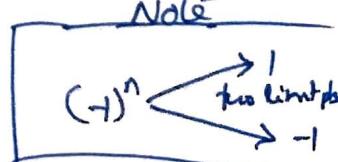
Remark The converse of Theorem 1 may not be true.

For example, consider the sequence

$$s_n = (-1)^n, n \in \mathbb{N}.$$

This sequence is bounded but not convergent.

If possible, let  $s_n \rightarrow l$ . Then for  $\epsilon = 1$ ,  $\exists m \in \mathbb{N}$  such that  $|s_n - l| < 1 \forall n \geq m$ , that is,  $|(-1)^n - l| < 1 \forall n \geq m$ . That is, in particular for  $n = 2m$ , we have  $|(-1)^{2m} - l| < 1$  and for  $n = 2m+1$ , we have  $|(-1)^{2m+1} - l| < 1$ . Consequently,



we have

$$|l-l'| < 1 \text{ and } |l+l'| < 1.$$

On the other hand, by triangular inequality, we have

$$\begin{aligned} 2 &= |l-l| + |l+l'| < |l-l| + |l+l'| \\ &\leq |l-l'| + |l+l'| < 1+1 = 2, \end{aligned}$$

a contradiction.

Hence the converse of Theorem 1 is not necessarily true.

Theorem 2: A sequence cannot converge to more than one limit.

Proof: If possible, let there exists a sequence  $\{s_n\}$  converges to two different limits  $l$  and  $l'$ .

$$\text{Set } \epsilon = \frac{1}{2}|l-l'| > 0.$$

Since the sequence  $\{s_n\}$  converges to  $l$  and  $l'$ , there exist positive integers (natural numbers)  $m$  and  $m'$  such that

$$|s_n - l| < \epsilon \text{ for } n \geq m$$

$$\text{and } |s_n - l'| < \epsilon \text{ for } n \geq m'.$$

~~Set~~ Hence for  $n \geq m = \max\{m, m'\}$ , we have

$$|s_n - l| < \epsilon \text{ and } |s_n - l'| < \epsilon \text{ for } n \geq m,$$

consequently, we have

$$\begin{aligned} |l - l'| &= |l - s_n + s_n - l'| \leq |l - s_n| + |s_n - l'| \\ &= |s_n - l| + |s_n - l'| \\ &< \epsilon + \epsilon = 2\epsilon = |l - l'|, \end{aligned}$$

a contradiction.

Thus, the sequence  $\{s_n\}$  cannot converge to two limits. The theorem is proved.

Remark: We have observed that a sequence converges to a number which is a limit point of the sequence. This limit point is unique. We say, sometimes, as the limit of the sequence, and symbolically, we write/ express as  $\lim_{n \rightarrow \infty} s_n = l$  or  $s_n \rightarrow l$  as  $n \rightarrow \infty$  or  $\lim s_n = l$ .

The above two theorems give the following beautiful criteria.

Theorem 3. Every convergent sequence is bounded and has a limit point.

### Limit Points of a Sequence

A real number  $\xi$  is said to be a limit point of a sequence  $\{s_n\}$  if every nbhd. of  $\xi$  contains an infinite number of members of the sequence.

Then,  $\xi$  is a limit point of the sequence if given any positive number  $\epsilon$ , however small,

$$s_n \in (\xi - \epsilon, \xi + \epsilon)$$

for an infinite values of  $n$ , i.e.,

$$|s_n - \xi| < \epsilon \text{ for infinite no. of values of } n.$$

In other words,  $s_n$  is arbitrary close to  $\xi$  for an infinite number of values of  $n$  or infinite no. of members of the sequence are very close to  $\xi$ .

Note: A number  $\xi$  is not a limit point of the sequence  $\{s_n\}$  if there exists a number  ~~$\epsilon$~~   $\epsilon > 0$  such that  $s_n \in (\xi - \epsilon, \xi + \epsilon)$  for at most a finite no. of values of  $n$ .

### Some Examples

(1)  $\{s_n\}$ ,  $s_n = 1 + n$ . Constant sequence.

This has only ~~one~~ one limit point 1.  
However, the range set in  $\{1\}$  which has no limit point.

(2)  $\{s_n\}$ ,  $s_n = \frac{1}{n}$ ,  $n \in \mathbb{N}$ . 0 is the limit point,  
which is also a limit point of the range  
 $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ .

(3).  $\{s_n\}$ ,  $s_n = 1 + (-1)^n$ ,  $n \in \mathbb{N}$ . 0 and 2 are the limit  
points. However, the range set  $\{0, 2\}$  has  
no limit points.

(4)  $\{s_n\}$ ,  $s_n = (-1)^n$ ,  $n \in \mathbb{N}$ . 1, -1 are the limit points.  
However, the range set  $\{-1, 1\}$  has no limit points.

(5-).  $\{s_n\} = (-1)^n(1 + \frac{1}{n})$ ,  $n \in \mathbb{N}$ . 1, -1 are the limit points  
which are the limit points of ~~the~~ its range set.