

Open sets, closed sets and Countable sets

Neighbourhood (nbhd) of a point :

A set $N \subseteq \mathbb{R}$ is called a nbhd. of a point 'a', if there exists an open interval I containing 'a' and contained in N , that is,

$$a \in I \subseteq N.$$

→ an open interval is a nbhd. of each of its points.

→ denote: $(a-\delta, a+\delta)$ to be a nbhd. of 'a', for some $\delta > 0$.

→ the set $\{x; 0 < |x-a| < \delta\}$, i.e. an open interval $(a-\delta, a+\delta)$ from which the number "a" itself has been excluded/deleted is called a deleted nbhd. of 'a'.

Some Examples .

1. The set \mathbb{R} of real numbers is the nbhd. of each of its points.

2. The set \mathbb{Q} of rational numbers is not a nbhd. of any of its points.

3. (a, b) is a nbhd. of each of its points.

4. $[a, b]$ is a nbhd. of each of points of (a, b) but not for 'a' & 'b' (being the end points).

5. The empty set \emptyset is a nbhd. of each of its points.

6. Super set of a nbhd. of a point 'a' is also a nbhd. of 'a'.

7. Union of finite/arbitrary nbhds. of a point 'x' is also a nbhd.

However, intersection of finite no. of nbhd. of a point x is a nbhd. of x .

Interior points of a set

A point x is an interior point of a set S if S is a nbhd. of x .

$\xrightarrow{\text{In other words}}$ x is an interior point of S if \exists an open interval (a, b) of x such that $x \in (a, b) \subseteq S$.

\rightarrow The set of all interior points of a set is called its interior of the set. denote S^i or $\text{Int } S$.

Remark: $\text{Int. } \mathbb{R} = \mathbb{R}$.

$\text{Int. } \mathbb{N} = \emptyset$, $\text{Int. } \mathbb{I} = \emptyset$, $\text{Int. } \mathbb{Q} = \emptyset$.

Homework: Prove that $S^i \subseteq S$.

Open set: A set S is said to be open if it is a nbhd. of each of its points, i.e., $\forall x \in S$, $\exists N(\exists z \downarrow \text{nbhd})$ such that $x \in N \subseteq S$.

\rightarrow Thus, every point of an open set is an interior point. So, for any open set S , we have $S = S^i$ or $\text{Int } S$.

Hence S is open iff $S = \text{Int } S$.

\rightarrow All the above discussion fails if \exists a point in S which is not an interior point of S , i.e., a set S is not open if it is not a nbhd. of at least one of its points, or, there exists at least one element in S which is not an interior point.