

Order Structure :

A field S is an ordered field if it satisfies the following properties:

O-1. Law of Trichotomy : For any two elements $a, b \in S$, only one of the following is true:
 $a > b, a = b, a < b.$

O-2 Transitivity : If $a, b, c \in S$,
 $a > b \wedge b > c \Rightarrow a > c.$

O-3 Compatibility of Order Relation with Addition Composition:
 $\forall a, b, c \in S, a > b \Rightarrow a + c > b + c.$

O-4 Compatibility of order relation with multiplication Composition:
 $\forall a, b, c \in S, a > b \wedge c > 0 \Rightarrow ac > bc.$

$\mathbb{Q}, \mathbb{R} \rightarrow$ order fields.

$\mathbb{N}, \mathbb{I} \rightarrow$ not fields.

The Set N of Natural Numbers:

We begin ~~the~~ the development of real numbers with the set N of natural numbers : $1, 2, 3, \dots$

Axioms: (Peano Postulates).: Properties of natural numbers.

P₁. $1 \in N$; i.e., N is a nonempty set and contains an element we designate as 1 ;

P₂. For each $n \in N$, there exists a unique element $n_0 \in N$ called the successor of n ;

P₃. For each $n \in N$, $n_0 \neq 1$; that n_0 is the successor of any element in N ;

P₄. For each pair $n, m \in N$ with $n \neq m$, $n_0 \neq m_0$; that is, distinct elements in N have distinct successors;

P₅. If $A \subseteq N$, $1 \in A$ and $b \in A$ implies $b_0 \in A$, then $A = N$.

↳ principle of Mathematical Induction.

As a consequence of $P_1 - P_r$, we have

{ Every nonempty subset of natural numbers
has a first element.
→ Well ordering principle in \mathbb{N} .

Intervals - Open and Closed

A subset A of \mathbb{R} is called an interval if A contains
(i) at least two distinct elements, and
(ii) every element lies between any two members of A .

Open Interval:

If a and b are two real numbers such that
 $a < b$, then the set

$$\{x; a < x < b\}$$

consisting of all real numbers between a and b
(excluding a and b) is called an open interval
and is denoted by (a, b) .

Closed Interval

The set

$$\{x; a \leq x \leq b\}$$

consisting of a and b and all real numbers between
 a and b is called a closed interval, and is denoted
by $[a, b]$.

Semi-closed or Semi-open Interval:

$$(a, b] = \{x; a < x \leq b\}$$

$$[a, b) = \{x; a \leq x < b\}.$$

Bounded and Unbounded Sets: Supremum & Infimum:

A subset S of real numbers is said to be bounded above if there exists a real number K such that every member of S is less than or equal to K , i.e.,

$$x \leq K \quad \forall x \in S.$$

$K \rightarrow$ upper bound of S .

- If no such K exists, then the set is said to be unbounded above or not bounded above.

The set S is said to be bounded below if there exists a real number k such that every member of S is greater than or equal to k , i.e.,

$$k \leq x \quad \forall x \in S.$$

$k \rightarrow$ lower bound of S .

- If no such k exists, then the set is said to be bounded below or not bounded below.

A set is said to be bounded if it is bounded above as well as below.

- If a set has one upper bound, it has an infinite number of upper bounds.

For example, if K is an upper bound, then every number greater than K is also an upper bound of S .

- Thus, every set S bounded above determines an infinite set of upper bounds.

This set of upper bounds is bounded below in as much as every member of S is ~~a~~ a lower bound thereof.

Similarly, a set S bounded below determines an infinite set of its lower bounds, which is bounded above by a member of S .

The above discussions leads us to the following:

Important: A member G of a set S is called the greatest member of S if every member of S is less than or equal to G , i.e.,

- (i) $G \in S$
- (ii) $x \leq G \quad \forall x \in S$.

Similarly, a member g of the set is called its smallest (or the least) member if every member of the set is greater than or equal to g .

Clearly, a set may or may not have the greatest or least member but an upper (lower) bound of the set, if it is a member of the set, is its greatest (least) member. A finite set always has the greatest as well as the smallest member.

- . If the set of all upper bounds of a set has the smallest member, say M , then M is called the least upper bound (l.u.b) or the supremum of S .
- . Clearly, the supremum of a set S may or may not exist and in case it exists, it may or may not belong to S . The fact that supremum M is the smallest of all the upper bounds of S . This may be described by the following two properties:

(i) M is the upper bound of S , i.e.,

$$x \leq M \quad \forall x \in S,$$

(ii) No member less than M can be an upper bound of S , i.e., for any positive number ϵ , however small, there is a number $y \in S$ s.t.

$$y > M - \epsilon.$$

- Again, it may be seen that a set cannot have more than one supremum.

for, let if possible M and M' be two suprema of a set, so that M and M' are both upper bounds of S .

since M is a l.u.b and M' is an upper bound of S ,

then $M \leq M'$.

Again, M' is the l.u.b and M is an upper bound of S ,

then $M' \leq M$.

Hence $M = M'$.

- if the set of all lower bounds of a set S has the greatest member, say m , then ~~m~~ m is called the greatest lower bound (g.l.b) or the infimum of S .

- Like the supremum, the infimum of a set may or may not exist and it may or may not belong to S . As above (just before we proved) it can be shown that a set cannot have more than one infimum.

The infimum m of a set S has the following two properties :

(i) m is the lower bound of S , i.e., $m \leq x \quad \forall x \in S$.

(ii) No member greater than m can be a lower bound of S , i.e., for any positive number ϵ ,

however small, there exists a number ~~such~~ $z \in S$ such that
 $z < m + \epsilon$.

Illustrations :

1. The set of natural numbers N is bounded below but not bounded above, \leftarrow lower bound.
2. The set I, Q and R are not bounded.
3. Every finite set of numbers is bounded.
4. The set S_1 of all positive real numbers
$$S_1 = \{x : x > 0, x \in \mathbb{R}\}$$
is not bounded above, but it is bounded below. The infimum 0 is not a member of the set S .
5. The infinite set $S_2 = \{x : 0 < x < 1, x \in \mathbb{R}\}$ is bounded with supremum 1 and infimum 0, both of which do not belong to S_2 .
6. The infinite set $S_3 = \{x : 0 \leq x \leq 1, x \in \mathbb{Q}\}$ is bounded with supremum 1 and infimum 0 which are the members of S .
7. The set $S_4 = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ is bounded. The supremum 1 belongs to S_4 while the infimum 0 does not.
8. Each of the following intervals is bounded:
 $(a, b), [a, b], (a, b], [a, b)$.

Home Work

Prove that the greatest member of a set, if it exists, is the supremum (l.u.b) of the set.