

If  $f: A \rightarrow B$  is both one-one and onto, then we say that  $f$  is a one to one correspondence between  $A$  and  $B$ . In this case,  $f^{-1}: B \rightarrow A$  is also a one to one correspondence between  $B$  and  $A$ .

If  $A_1 \subseteq A$ , then its image  $f(A_1)$  is a subset of  $B$ , defined by

$$f(A_1) = \{ f(x) \in B; x \in A_1 \}.$$

Similarly, if  $B_1$  is a subset of  $B$ , then its inverse image  $f^{-1}(B_1)$  is a subset of  $A$ , defined by

$$f^{-1}(B_1) = \{ x \in A; f(x) \in B_1 \}.$$

A function  $f$  is called an extension of a function  $g$  (and  $g$  is a restriction of  $f$ ) if the domain of  $f$  contains the domain of  $g$  and

$$f(x) = g(x) \text{ for each } x \text{ in the domain of } g.$$

We can combine functions in the following way:

$f: A \rightarrow B$ ,  $g: B \rightarrow C$ , then composition  $\circ$  function  $g \circ f: A \rightarrow C$  is given by

$$(g \circ f)(x) = g(f(x)) \quad \forall x \in A.$$

$I: A \rightarrow A$  defined by  $I(x) = x$  (identity function on  $A$ )

If  $g \circ f = f \circ g = I$ , then  $g = f^{-1}$  or  $f = g^{-1}$ .

$\mathbb{N} \rightarrow$  set of Natural numbers

$\mathbb{I} \rightarrow$  set of Integers

$\mathbb{Q} \rightarrow$  set of rationals

$\mathbb{Q}' \rightarrow$  set of irrationals

$\mathbb{R} \rightarrow$  set of Real numbers.

$$\mathbb{N} \subset \mathbb{I} \subset \mathbb{Q}, \quad \mathbb{Q}, \mathbb{Q}' \subseteq \mathbb{R}.$$

Compositions:

- Addition Composition: If  $+$  is defined on a set  $S$  if to each pair of members  $a, b$  of  $S$ , there corresponds a member  $a+b$  of  $S$ .

2. Multiplication Composition is defined on a set  $S$  if to each pair of members  $a, b$  of  $S$ , there corresponds a ~~member~~ member  $ab$  of  $S$ .

If on a set, we define the addition & multiplication composition, then the set possesses an algebraic structure.

Subtraction: Let  $a, b \in S$ .  $a - b = a + (-b)$ . When  $-b \in S$ .

Division: ~~Let~~ The quotient  $a/b$  ( $b \neq 0$ ) may be put as  $a \cdot \frac{1}{b}$  or  $ab^{-1}$  when  $\frac{1}{b}$  or  $b^{-1} \in S$ .

Now, we are ready to go the main syllabus.

# Field Structure and Order Structure

## Field structure.

A set  $S$  is said to be a field if two compositions of addition and multiplication defined in it be such that  $\forall a, b, c \in S$ , the following properties are satisfied

A-1 Set  $S$  is closed under addition:

$$a, b \in S \Rightarrow a + b \in S.$$

A-2 Addition is commutative:  $a + b = b + a.$

A-3 Addition is associative:  $(a + b) + c = a + (b + c)$

A-4 Additive identity exists, i.e.,  $\exists$  a member  $0 \in S$  such that  $a + 0 = a.$

A-5 Additive inverse exists, i.e., to each element  $a \in S$ , there exists an element  $-a \in S$  s.t.

$$a + (-a) = 0.$$

M-1.  $S$  is closed under multiplication:

$$a, b \in S \Rightarrow ab \in S.$$

M-2 Multiplication is commutative:  $ab = ba.$

M-3 Multiplication is associative:  $(ab)c = a(bc).$

M-4 Multiplicative identity exists, i.e.,  $\exists$  a member  $1 \in S$  s.t.  $a \cdot 1 = a.$

M-5 Multiplicative inverse exists, i.e., to each  $a (\neq 0) \in S$ ,  $\exists$  an element  $a^{-1} \in S$  s.t.  $aa^{-1} = 1.$

AM. Multiplication is distributive w.r. to addition, i.e.,  $a(bc) = ab + ac.$

Thus, a set  $S$  has a field structure if it possesses the two compositions of addition and multiplication and satisfies the eleven properties listed above.