

# The Riemann-Stieltjes Integral

Having discussed the Riemann theory of integration to the extent possible within the scope of the present discussion, we now pass on to a generalisation of the subject. As mentioned earlier many refinements and extensions of the theory exist but we shall study briefly—in fact very briefly—the extension due to Stieltjes, known as the theory of *Riemann-Stieltjes integration*. The most noteworthy of the extensions, the *Lebesgue theory* of integration will be however discussed later in chapter 19.

It may be stated once for all that, unless otherwise stated, all functions will be *real-valued* and *bounded* on the domain of definition. The function  $\alpha$  will always be *monotonic increasing*.

## 1. DEFINITIONS AND EXISTENCE OF THE INTEGRAL

Let  $f$  and  $\alpha$  be bounded function on  $[a, b]$  and  $\alpha$  be *monotonic increasing* on  $[a, b]$ ,  $b \geq a$ .

Corresponding to any partition

$$P = \{a = x_0, x_1, \dots, x_n = b\}, \text{ of } [a, b]$$

we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}), \quad i = 1, 2, \dots, n.$$

It is clear that  $\Delta\alpha_i \geq 0$ . As in § 1.1 Ch. 9, we define two sums,

$$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i$$

$$L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i$$

where  $m_i, M_i$  are the bounds (infimum and supremum respectively) of  $f$  in  $\Delta x_i$ , respectively called the *Upper* and the *Lower* sums of  $f$  corresponding to the partition  $P$ .

If  $m, M$  are respectively the lower and the upper bounds of  $f$  on  $[a, b]$ , we have

$$m \leq m_i \leq M_i \leq M$$

$$\Rightarrow m \Delta\alpha_i \leq m_i \Delta\alpha_i \leq M_i \Delta\alpha_i \leq M \Delta\alpha_i, \quad \Delta\alpha_i \geq 0$$

Putting  $i = 1, 2, \dots, n$  and adding all inequalities, we get

$$m\{\alpha(b) - \alpha(a)\} \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M\{\alpha(b) - \alpha(a)\} \quad (1)$$

As in Riemann integration, § 1.1, we define two integrals, which always exist by a similar reasoning,

$$\int_a^b f d\alpha = \inf . U(P, f, \alpha)$$

$$\int_a^b f d\alpha = \sup . L(P, f, \alpha) \quad (2)$$

the infimum and supremum being taken over all partitions of  $[a, b]$ . These are respectively called the *upper* and the *lower* integrals of  $f$  with respect to  $\alpha$ .

These two integrals may or may not be equal. In cases these two integrals are equal, i.e.,

$$\int_a^b f d\alpha = \int_a^b f d\alpha,$$

we say that  $f$  is integrable with respect to  $\alpha$  in the Riemann sense and write  $f \in \mathcal{R}_\alpha [a, b]$  or simply  $\mathcal{R}(\alpha)$ . Their common value is denoted by

$$\int_a^b f d\alpha$$

or sometimes by

$$\int_a^b f(x) d\alpha(x)$$

and is called the *Riemann-Stieltjes integral* (or simply the *Stieltjes integral*) of  $f$  with respect to  $\alpha$ , over  $[a, b]$ .

From (1) and (2), it follows that

$$m\{\alpha(b) - \alpha(a)\} \leq L(P, f, \alpha) \leq \int_a^b f d\alpha \leq \int_a^b f d\alpha$$

$$\leq U(P, f, \alpha) \leq m\{\alpha(b) - \alpha(a)\} \quad (3)$$

**Remark.** The upper and the lower integrals always exist for bounded functions but these may not be equal for all bounded functions. Such functions are not integrable. Thus the question of their equality and hence that of the integrability of the function is our main concern.

The Riemann-Stieltjes integral reduces to Riemann integral when  $\alpha(x) = x$ .

## 1 Some Deductions

(i) If  $f \in \mathcal{R}(\alpha)$ , then  $\exists$  a number  $\lambda$  lying between the bounds of  $f$  such that

$$\int_a^b f d\alpha = \lambda\{\alpha(b) - \alpha(a)\} \quad (\text{using 3})$$

(ii) If  $f$  is continuous on  $[a, b]$ , then  $\exists$  a number  $\xi \in [a, b]$  such that

$$\int_a^b f d\alpha = f(\xi) \{\alpha(b) - \alpha(a)\}$$

(iii) If  $f \in \mathcal{R}(\alpha)$ , and  $k$  is a number such that

$$|f(x)| \leq k, \text{ for all } x \in [a, b]$$

then

$$\left| \int_a^b f d\alpha \right| \leq k \{ \alpha(b) - \alpha(a) \}$$

(iv) If  $f \in \mathcal{R}(\alpha)$  over  $[a, b]$  and  $f(x) \geq 0$ , for all  $x \in [a, b]$ , then

$$\int_a^b f d\alpha \begin{cases} \geq 0, & b \geq a \\ \leq 0, & b \leq a \end{cases}$$

Since  $f(x) \geq 0$ , the lower bound  $m \geq 0$  and therefore the result follows from (3).

(v) If  $f \in \mathcal{R}(\alpha)$ ,  $g \in \mathcal{R}(\alpha)$  over  $[a, b]$  such that  $f(x) \geq g(x)$ , then

$$\int_a^b f d\alpha \geq \int_a^b g d\alpha, \quad b \geq a,$$

and

$$\int_a^b f d\alpha \leq \int_a^b g d\alpha, \quad b \leq a$$

The result follows by reasoning similar to that of Deduction 5 § 1.4, Chapter 9.

## 1.2 Refinement of Partitions

**Theorem 1.** If  $P^*$  is a refinement of  $P$ , then

(i)  $L(P^*, f, \alpha) \geq L(P, f, \alpha)$ , and

(ii)  $L(P^*, f, \alpha) \leq U(P, f, \alpha)$ .

Let us prove (ii).

Let  $P = \{a = x_0, x_1, \dots, x_n = b\}$  be a partition of the given interval. Suppose first that  $P^*$  contains just one point more than  $P$ . Let this extra point  $\xi$  belong to  $\Delta x_i$ , i.e.,  $x_{i-1} < \xi < x_i$ .

As  $f$  is bounded over the entire interval  $[a, b]$ , it is bounded on every sub-interval  $\Delta x_i$  ( $i = 1, 2, \dots, n$ ).

Let  $W_1, W_2, M_i$  be the upper bounds (supremum) of  $f$  in the intervals  $[x_{i-1}, \xi], [\xi, x_i], [x_{i-1}, x_i]$ , respectively.

Clearly

$$W_1 \leq M_i, \quad W_2 \leq M_i.$$

$$\begin{aligned} \therefore U(P^*, f, \alpha) - U(P, f, \alpha) &= W_1 \{ \alpha(\xi) - \alpha(x_{i-1}) \} + W_2 \{ \alpha(x_i) - \alpha(\xi) \} \\ &\quad - M_i \{ \alpha(x_i) - \alpha(x_{i-1}) \} \end{aligned}$$

$$= (W_1 - M_i) \{a(\xi) - \alpha(x_{i-1})\} + (W_2 - M_i) \{a(x_i) - \alpha(\xi)\} \leq 0$$

$$\Rightarrow U(P^*, f, \alpha) \leq U(P, f, \alpha)$$

If  $P^*$  contains  $m$  points more than  $P$ , we repeat the above reasoning  $m$  times and arrive at the result

(ii).

The proof of (i) is similar.

**Theorem 2.** For any two partitions  $P_1, P_2$ ,

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

i.e., no upper sum can ever be less than any lower sum.

**Corollary.** For a bounded function  $f$ ,

$$\int_a^b f d\alpha \leq \int_a^b f d\alpha$$

The proofs are similar to that of Theorem 2 Chapter 9.

**Ex.** If  $P^* \supseteq P$ , then show that

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha)$$

## 2. A CONDITION OF INTEGRABILITY

**Theorem 3.** A function  $f$  is integrable with respect to  $\alpha$  on  $[a, b]$  if and only if for every  $\varepsilon > 0$  there exists a partition  $P$  of  $[a, b]$  such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

*Necessary.* Let  $f \in \mathcal{R}(\alpha)$  over  $[a, b]$

$\therefore$

$$\int_a^b f d\alpha = \bar{\int}_a^b f d\alpha = \int_a^b f d\alpha$$

Let  $\varepsilon > 0$  be any number.

Since the upper and the lower integrals are the infimum and the supremum, respectively, of the upper and the lower sums, therefore  $\exists$  partitions  $P_1$  and  $P_2$  such that

$$U(P_1, f, \alpha) < \bar{\int}_a^b f d\alpha + \frac{1}{2}\varepsilon = \int_a^b f d\alpha + \frac{1}{2}\varepsilon$$

$$L(P_2, f, \alpha) > \int_a^b f d\alpha - \frac{1}{2}\varepsilon = \int_a^b f d\alpha - \frac{1}{2}\varepsilon$$

Let  $P = P_1 \cup P_2$  be the common refinement of  $P_1$  and  $P_2$ .

$\therefore$

$$U(P, f, \alpha) \leq U(P_1, f, \alpha)$$

$$< \int_a^b f d\alpha + \frac{1}{2}\varepsilon < L(P_2, f, \alpha) + \varepsilon \leq L(P, f, \alpha) + \varepsilon$$

$$\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

Sufficient. For  $\varepsilon > 0$ , let  $P$  be a partition for which

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

For any partition  $P$ , we know that

$$L(P, f, \alpha) \leq \int_a^b f d\alpha \leq \bar{\int}_a^b f d\alpha \leq U(P, f, \alpha)$$

$$\therefore \quad \bar{\int} f d\alpha - \int f d\alpha \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

But a non-negative number can be less than every positive number, if it is zero.

$$\therefore \quad \bar{\int}_a^b f d\alpha - \int_a^b f d\alpha = 0$$

so that  $f \in \mathcal{R}(\alpha)$ , over  $[a, b]$ .

### 3. SOME THEOREMS

(a) If  $f_1 \in \mathcal{R}(\alpha)$  and  $f_2 \in \mathcal{R}(\alpha)$  over  $[a, b]$ , then

$$f_1 + f_2 \in \mathcal{R}(\alpha) \text{ and } \int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

(b) If  $f \in \mathcal{R}(\alpha)$ , and  $c$  is a constant, then

$$cf \in \mathcal{R}(\alpha) \text{ and } \int_a^b cf d\alpha = c \int_a^b f d\alpha$$

(c) If  $f_1 \in \mathcal{R}(\alpha)$ ,  $f_2 \in \mathcal{R}(\alpha)$  and  $f_1(x) \leq f_2(x)$  on  $[a, b]$  then

$$\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$$

(d) If  $f \in \mathcal{R}(\alpha)$  over  $[a, b]$  and if  $a < c < b$ , then

$$f \in \mathcal{R}(\alpha) \text{ on } [a, c], \text{ and on } [c, b]$$

$$\text{and } \int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$$

(e) If  $f \in \mathcal{R}(\alpha)$  over  $[a, b]$ , then

$$|f| \in \mathcal{R}(\alpha) \text{ and } \left| \int_a^b f d\alpha \right| \leq \int_a^b |f| d\alpha$$

(f) If  $f \in \mathcal{R}(\alpha)$  on  $[a, b]$  then

$$f^2 \in \mathcal{R}(\alpha)$$

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(g) If  $f \in \mathcal{R}(\alpha_1)$  and  $f \in \mathcal{R}(\alpha_2)$ , then  
 $f \in \mathcal{R}(\alpha_1 + \alpha_2)$  and  $\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$

and if  $f \in \mathcal{R}(\alpha)$  and  $c$  a positive constant, then

$$f \in \mathcal{R}(c\alpha) \text{ and } \int_a^b f d(c\alpha) = c \int_a^b f d\alpha.$$

(a) Let  $f = f_1 + f_2$ .

Clearly  $f$  is bounded on  $[a, b]$ .

If  $P = \{a = x_0, x_1, \dots, x_n = b\}$  be any partition of  $[a, b]$  and  $m'_i, M'_i; m''_i, M''_i; m_i, M_i$  the bounds of  $f_1, f_2$  and  $f$ , respectively, on  $\Delta x_i$ , then

$$m'_i + m''_i \leq m_i \leq M_i \leq M'_i + M''_i$$

Multiplying by  $\Delta x_i$  and adding all these inequalities for  $i = 1, 2, 3, \dots, n$ .

we get

$$\begin{aligned} L(P, f_1, \alpha) + L(P, f_2, \alpha) &\leq L(P, f, \alpha) \leq U(P, f, \alpha) \\ &\leq U(P, f_1, \alpha) + U(P, f_2, \alpha) \end{aligned} \quad (1)$$

Let  $\varepsilon > 0$  be any number.

Since  $f_1 \in \mathcal{R}(\alpha), f_2 \in \mathcal{R}(\alpha)$ , therefore  $\exists$  partitions  $P_1, P_2$  such that

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{1}{2}\varepsilon$$

$$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{1}{2}\varepsilon$$

Let  $P = P_1 \cup P_2$ , a refinement of  $P_1$  and  $P_2$ .

$\therefore$

$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \frac{1}{2}\varepsilon$$

$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \frac{1}{2}\varepsilon \quad (2)$$

Thus for partition  $P$ , we get from (1) and (2),

$$U(P, f, \alpha) - L(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_1, \alpha) - L(P, f_2, \alpha)$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

$$\Rightarrow f \in \mathcal{R}(\alpha) \text{ over } [a, b]$$

Let us now proceed to prove the second part.

Since the upper integral is the infimum of the upper sums, therefore  $\exists$  partitions  $P_1, P_2$  such that

$$U(P_1, f_1, \alpha) < \int_a^b f_1 d\alpha + \frac{1}{2}\varepsilon$$

$$U(P_2, f_2, \alpha) < \int_a^b f_2 d\alpha + \frac{1}{2}\varepsilon$$

If  $P = P_1 \cup P_2$ , we have

$$\left. \begin{aligned} U(P, f_1, \alpha) &< \int_a^b f_1 d\alpha + \frac{1}{2} \varepsilon \\ U(P, f_2, \alpha) &< \int_a^b f_2 d\alpha + \frac{1}{2} \varepsilon \end{aligned} \right\} \quad (3)$$

For such a partition  $P$ ,

$$\begin{aligned} \int_a^b f d\alpha &\leq U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha) \quad [\text{from (1)}] \\ &\leq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + \varepsilon \quad [\text{using (3)}] \end{aligned}$$

Since  $\varepsilon$  is arbitrary, we get

$$\int_a^b f d\alpha \leq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \quad (4)$$

Proceeding with  $(-f_1)$  and  $(-f_2)$  instead of  $f_1$  and  $f_2$ , we get

$$\int_a^b f d\alpha \geq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \quad (5)$$

(4) and (5) give

$$\int_a^b f d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

(g) Since  $f \in \mathcal{R}(\alpha_1)$  and  $f \in \mathcal{R}(\alpha_2)$ , therefore for  $\varepsilon > 0$ ,  $\exists$  partitions  $P_1, P_2$  of  $[a, b]$  such that

$$\begin{aligned} U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) &< \frac{1}{2} \varepsilon \\ U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) &< \frac{1}{2} \varepsilon \end{aligned}$$

Let  $P = P_1 \cup P_2$

$$\therefore U(P, f, \alpha_1) - L(P, f, \alpha_1) < \frac{1}{2} \varepsilon \quad (1)$$

$$U(P, f, \alpha_2) - L(P, f, \alpha_2) < \frac{1}{2} \varepsilon$$

Let the partition  $P$  be  $\{a = x_0, x_1, x_2, \dots, x_n = b\}$ , and  $m_i, M_i$  be bounds of  $f$  in  $\Delta x_i$ .

Let  $\alpha = \alpha_1 + \alpha_2$ .

$$\therefore \alpha(x) = \alpha_1(x) + \alpha_2(x)$$

$$\Delta x_{ii} = \alpha_1(x_i) - \alpha_1(x_{i-1})$$

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$$\Delta x_{2i} = \alpha_2(x_i) - \alpha_2(x_{i-1})$$

$$\Delta x_i = \alpha(x_i) - \alpha(x_{i-1})$$

 $\therefore$ 

$$\begin{aligned} &= \alpha_1(x_i) + \alpha_2(x_i) - \alpha_1(x_{i-1}) - \alpha_2(x_{i-1}) \\ &= \Delta \alpha_{1i} + \Delta \alpha_{2i} \end{aligned}$$

 $\therefore$ 

$$\begin{aligned} U(P, f, \alpha) &= \sum_i M_i \Delta \alpha_i \\ &= \sum_i M_i (\Delta \alpha_{1i} + \Delta \alpha_{2i}) \\ &= U(P, f, \alpha_1) + U(P, f, \alpha_2) \end{aligned} \quad (2)$$

Similarly,

$$L(P, f, \alpha) = L(P, f, \alpha_1) + L(P, f, \alpha_2) \quad (3)$$

 $\therefore$ 

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= U(P, f, \alpha_1) - L(P, f, \alpha_1) \\ &\quad + U(P, f, \alpha_2) - L(P, f, \alpha_2) \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \text{ [using (1)]} \end{aligned}$$

$$\Rightarrow f \in \mathcal{R}(\alpha), \text{ where } \alpha = \alpha_1 + \alpha_2$$

Now to prove the second part, we notice that

$$\begin{aligned} \int_a^b f d\alpha &= \inf U(P, f, \alpha) \\ &= \inf \{U(P, f, \alpha_1) + U(P, f, \alpha_2)\} \\ &\geq \inf U(P, f, \alpha_1) + \inf U(P, f, \alpha_2) \\ &= \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \end{aligned} \quad (4)$$

Similarly,

$$\begin{aligned} \int_a^b f d\alpha &= \sup L(P, f, \alpha) \\ &\leq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \end{aligned} \quad (5)$$

From (4) and (5),

$$\int_a^b f d\alpha = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

where  $\alpha = \alpha_1 + \alpha_2$ .

The proofs of the remaining parts are so similar to the above proofs and virtually identical to those of the corresponding theorems for Riemann integral that it is a mere repetition and are therefore left to the reader.

**Corollary.** If  $f_1 \in \mathcal{R}(\alpha)$  and  $f_2 \in \mathcal{R}(\alpha)$  over  $[a, b]$ , then

$$f_1 \cdot f_2 \in \mathcal{R}(\alpha)$$

We know that if  $f_1, f_2$  are integrable then  $f_1 + f_2, f_1 - f_2, f_1^2, f_2^2$ , are all integrable.

Also, then  $(f_1 + f_2)^2, (f_1 - f_2)^2$  are integrable.

Now

$$\begin{aligned} 4f_1 \cdot f_2 &= (f_1 + f_2)^2 - (f_1 - f_2)^2 \\ \Rightarrow f_1 \cdot f_2 &\in \mathcal{R}(\alpha) \end{aligned}$$

#### 4. A DEFINITION (Integral as a limit of sum)

As an analog to the Riemann sum, we introduce a sum which will lead to a sufficient condition for the existence of a Riemann-Stieltjes integral.

**Definition.** Corresponding to a partition  $P$  of  $[a, b]$  and  $t_i \in \Delta x_i$ , consider the sum

$$S(P, f, \alpha) = \sum_{i=1}^n f(t_i) \Delta \alpha_i$$

We say that  $S(P, f, \alpha)$  converges to  $A$  as  $\mu(P) \rightarrow 0$ , i.e.,

$$\lim_{\mu(P) \rightarrow 0} S(P, f, \alpha) = A$$

if, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|S(P, f, \alpha) - A| < \varepsilon$ , for every partition  $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ , of  $[a, b]$ , with mesh  $\mu(P) < \delta$  and every choice of  $t_i$  in  $\Delta x_i$ .

**Theorem 4.** If  $\lim S(P, f, \alpha)$  exists as  $\mu(P) \rightarrow 0$ , then

$$f \in \mathcal{R}(\alpha), \text{ and } \lim_{\mu(P) \rightarrow 0} S(P, f, \alpha) = \int_a^b f d\alpha$$

Let us suppose that  $\lim S(P, f, \alpha)$  exists as  $\mu(P) \rightarrow 0$  and is equal to  $A$ .

Therefore for  $\varepsilon > 0$ ,  $\exists \delta > 0$  such that for every partition  $P$  of  $[a, b]$  with mesh  $\mu(P) < \delta$  and every choice of  $t_i$  and  $\Delta x_i$ , we have

$$|S(P, f, \alpha) - A| < \frac{1}{2}\varepsilon$$

or

$$A - \frac{1}{2}\varepsilon < S(P, f, \alpha) < A + \frac{1}{2}\varepsilon \quad (1)$$

Let  $P$  be one such partition. If we let the points  $t_i$  range over the intervals  $\Delta x_i$  and take the infimum and the supremum of the sums  $S(P, f, \alpha)$ , (1) yields

$$\begin{aligned}
 A - \frac{1}{2}\varepsilon &< L(P, f, \alpha) \leq U(P, f, \alpha) < A + \frac{1}{2}\varepsilon \\
 \Rightarrow U(P, f, \alpha) - L(P, f, \alpha) &< \varepsilon \\
 \Rightarrow f &\in \mathcal{R}(\alpha) \text{ over } [a, b]
 \end{aligned}
 \tag{2}$$

Again, since  $S(P, f, \alpha)$  and  $\int_a^b f d\alpha$  lie between  $U(P, f, \alpha)$  and  $L(P, f, \alpha)$

$$\therefore \left| S(P, f, \alpha) - \int_a^b f d\alpha \right| \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

$$\Rightarrow \lim_{\mu(P) \rightarrow 0} S(P, f, \alpha) = \int_a^b f d\alpha$$

**Remark.** The theorem asserts that the existence of the limit of  $S(P, f, \alpha)$  implies that  $f \in \mathcal{R}(\alpha)$ . The existence of the limit is a sufficient condition for  $f \in \mathcal{R}(\alpha)$  but as shown in Example 3 it is not a necessary condition, i.e., functions exist which are integrable but for which limit of  $S(P, f, \alpha)$  does not exist. Thus whenever  $\lim S(P, f, \alpha)$  exists, it is equal to  $\int f d\alpha$ . But when  $f \in \mathcal{R}(\alpha)$  nothing can be said about the existence of  $\lim S(P, f, \alpha)$ .

**Theorem 5.** If  $f$  is continuous on  $[a, b]$  then  $f \in \mathcal{R}(\alpha)$  over  $[a, b]$ . Moreover, to every  $\varepsilon > 0$  there corresponds a  $\delta > 0$  such that

$$\left| S(P, f, \alpha) - \int_a^b f d\alpha \right| < \varepsilon$$

for every partition  $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$  of  $[a, b]$  with  $\mu(P) < \delta$ , and for every choice of  $t_i$  in  $\Delta x_i$ , i.e.,

$$\lim_{\mu(P) \rightarrow 0} S(P, f, \alpha) = \int_a^b f d\alpha$$

[We still assume that all functions are bounded and  $\alpha$  is monotonic increasing.]

Let  $\varepsilon > 0$  be given, and let us choose  $\eta > 0$  such that

$$\eta\{\alpha(b) - \alpha(a)\} < \varepsilon \tag{1}$$

Since continuity of  $f$  on the closed interval  $[a, b]$  implies its uniform continuity on  $[a, b]$  therefore for  $\eta > 0$  there corresponds  $\delta > 0$  such that

$$|f(t_1) - f(t_2)| < \eta, \text{ if } |t_1 - t_2| < \delta, t_1, t_2 \in [a, b] \tag{2}$$

Let  $P$  be a partition of  $[a, b]$ , with norm  $\mu(P) < \delta$ .

Then in view of (2),

$$\begin{aligned} \therefore \quad & M_i - m_i \leq \eta, \quad i = 1, 2, \dots, n \\ & U(P, f, \alpha) - L(P, f, \alpha) = \sum_i (M_i - m_i) \Delta x_i \\ & \leq \eta \sum_i \Delta x_i \\ & = \eta \{\alpha(b) - \alpha(a)\} < \varepsilon \\ \Rightarrow & f \in \mathcal{R}(\alpha) \text{ over } [a, b]. \end{aligned} \quad (3)$$

Again if  $f \in \mathcal{R}(\alpha)$ , then for  $\varepsilon > 0$ ,  $\exists \delta > 0$  such that for all partitions  $P$  with  $\mu(P) < \delta$ ,

$$|U(P, f, \alpha) - L(P, f, \alpha)| < \varepsilon$$

Since  $S(P, f, \alpha)$  and  $\int_a^b f d\alpha$  both lie between  $U(P, f, \alpha)$  and  $L(P, f, \alpha)$  for all partitions  $P$  with  $\mu(P) < \delta$  and for all positions of  $t_i$  in  $\Delta x_i$ .

$$\therefore \quad \left| S(P, f, \alpha) - \int_a^b f d\alpha \right| < U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

$$\Rightarrow \lim_{\mu(P) \rightarrow 0} S(P, f, \alpha) = \lim_{\mu(P) \rightarrow 0} \sum_{i=1}^n f(t_i) \Delta \alpha_i = \int_a^b f d\alpha$$

**Note 1.** Continuity is a sufficient condition for integrability of a function. It is not necessary condition. Functions exist which are integrable but not continuous.

**Note 2.** For continuous function  $f$ ,  $\lim S(P, f, \alpha)$  exists and equals  $\int f d\alpha$ .

**Theorem 6.** If  $f$  is monotonic on  $[a, b]$ , and if  $\alpha$  is continuous on  $[a, b]$ , then  $f \in \mathcal{R}(\alpha)$ .

[Monotonicity of  $\alpha$  is still assumed.]

Let  $\varepsilon > 0$  be a given positive number.

For any positive integer  $n$ , choose, a partition  $P = \{x_0, x_1, \dots, x_n\}$  of  $[a, b]$  such that

$$\Delta \alpha_i = \frac{\alpha(b) - \alpha(a)}{n}, \quad i = 1, 2, \dots, n$$

This is possible because  $\alpha$  is continuous and monotonic increasing on the closed interval  $[a, b]$  and thus assumes every value between its bounds,  $\alpha(a)$  and  $\alpha(b)$ .

Let  $f$  be monotonic increasing on  $[a, b]$ , so that its lower and the upper bound,  $m_i, M_i$  in  $\Delta x_i$  are given by

$$m_i = f(x_{i-1}), \quad M_i = f(x_i), \quad i = 1, 2, \dots, n$$

$$\begin{aligned}
 \therefore U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^n (M_i - m_i) \Delta x_i \\
 &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^n \{f(x_i) - f(x_{i-1})\} \\
 &= \frac{\alpha(b) - \alpha(a)}{n} \{f(b) - f(a)\} \\
 &< \varepsilon, \text{ for large } n \\
 \Rightarrow f &\in \mathcal{R}(\alpha) \text{ over } [a, b]
 \end{aligned}$$

**Note.**  $f \in \mathcal{R}(\alpha)$ , i.e.,  $\int f d\alpha$  exists when either

- (i)  $f$  is continuous and  $\alpha$  is monotonic, or
- (ii)  $f$  is monotonic and  $\alpha$  is continuous; of course  $\alpha$  is still monotonic.

## 4.1 Some Examples

**Example 1.** A function  $\alpha$  increases on  $[a, b]$  and is continuous at  $x'$  where  $a \leq x' \leq b$ . Another function  $f$  is such that

$$f(x') = 1, \text{ and } f(x) = 0, \text{ for } x \neq x'$$

Prove that

$$f \in \mathcal{R}(\alpha) \text{ over } [a, b], \text{ and } \int_a^b f d\alpha = 0$$

Let  $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$  be a partition of  $[a, b]$  and let  $x' \in \Delta x_i$ .

But since  $\alpha$  is continuous at  $x'$  and increases on  $[a, b]$ , therefore for  $\varepsilon > 0$  we can choose  $\delta > 0$  such that

$$\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1}) < \varepsilon, \text{ for } \Delta x_i < \delta$$

Let,  $P$  be a partition with  $\mu(P) < \delta$ . Now

$$U(P, f, \alpha) = \Delta \alpha_i$$

$$L(P, f, \alpha) = 0$$

$$\therefore \int_a^b f d\alpha = \inf U(P, f, \alpha), \text{ over all partitions } P \text{ with } \mu(P) < \delta$$

$$= 0 = \int_a^b f d\alpha$$

$$\Rightarrow f \in \mathcal{R}(\alpha), \text{ and } \int_a^b f d\alpha = 0.$$

**Aliter.** Let  $P = \{a = x_0, x_1, \dots, x_n = b\}$  be a partition of  $[a, b]$  and let  $x' \in \Delta x_i$ ,  $x_{i-1} \leq x' < x_i$ .

By continuity of  $\alpha$  at  $x'$ , for  $\varepsilon > 0$ ,  $\exists \delta > 0$  such that

$$|\alpha(x) - \alpha(x')| < \frac{1}{2}\varepsilon, \text{ for } |x - x'| < \delta$$

Again, since  $\alpha$  is an increasing function, we have

$$\alpha(x) - \alpha(x') < \frac{1}{2}\varepsilon, \text{ for } 0 < x - x' < \delta$$

and

$$\alpha(x') - \alpha(x) < \frac{1}{2}\varepsilon, \text{ for } 0 < x' - x < \delta$$

Let  $P$  be a partition with  $\mu(P) < \delta$ .

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

$\therefore$

$$= \alpha(x_i) - \alpha(x') + \alpha(x') - \alpha(x_{i-1})$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

$$S(P, f, \alpha) = \sum_{i=1}^n f(t_i) \Delta\alpha_i = f(t_i) \Delta\alpha_i$$

$$= \begin{cases} 0, & t_i \neq x' \\ \Delta\alpha_i, & t_i = x' \end{cases}$$

$\therefore$

$$|S(P, f, \alpha)| = 0, \text{ when } t_i \neq x' \\ < \varepsilon, \text{ when } t_i = x'$$

In either case

$$\lim_{\mu(P) \rightarrow 0} S(P, f, \alpha) = 0$$

$$\Rightarrow f \in \mathcal{R}(\alpha) \text{ over } [a, b], \text{ and } \int_a^b f d\alpha = 0.$$

**Example 2.**  $f$  is a function bounded on  $[-1, 1]$ , are three functions  $\beta_1, \beta_2, \beta_3$  are defined as follows:

$$\beta_1(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

$$\beta_2(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

$$\beta_3(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases}$$

Prove that  $f \in \mathcal{R}(\beta_3)$  iff  $f$  is continuous at  $x = 0$ , and then

$$\int_{-1}^1 f d\beta_3 = f(0)$$

Let  $P = \{-1 = x_0, x_1, \dots, x_{i-2}, 0 = x_{i-1}, x_i, \dots, x_n = 1\}$  be a partition of  $[-1, 1]$  such that  $x_{i-1} = 0$ . Let  $t_i \in \Delta x_i$ .

Now

$$\begin{aligned} S(P, f, \beta_3) &= \sum_{j=1}^n f(t_j) \{\beta_3(x_j) - \beta_3(x_{j-1})\} \\ &= f(t_{i-1}) \cdot \frac{1}{2} + f(t_i) \cdot (1 - \frac{1}{2}) \\ &= \frac{1}{2} \{f(t_{i-1}) + f(t_i)\} \end{aligned} \quad (1)$$

$$= f(0) \text{ in particular when } t_{i-1} = 0 = t_i \quad (2)$$

Clearly  $t_{i-1}$  tends to 0 from below and  $t_i$  from above, when the norm  $\mu(P)$  tends to zero.

Hence  $\lim_{\mu(P) \rightarrow 0} S(P, f, \alpha)$  exists when both the limits,  $\lim_{t_{i-1} \rightarrow 0} f(t_{i-1})$  and  $\lim_{t_i \rightarrow 0} f(t_i)$  or equivalently

$\lim_{x \rightarrow 0-0} f(x)$  and  $\lim_{x \rightarrow 0+0} f(x)$ , exist, i.e., both  $f(0-)$  and  $f(0+)$  exist.

Moreover, from (2) it is evident that these limits are each equal to  $f(0)$ . In that case

$$\lim_{\mu(P) \rightarrow 0} S(P, f, \alpha) = f(0)$$

Hence  $f \in \mathcal{R}(\beta_3)$  if  $f(0+) = f(0-) = f(0)$ , i.e., if the function  $f$  is continuous at zero and in that case

$$\int_{-1}^1 f d\beta_3 = f(0)$$

Also it is clear that  $f$  is continuous if  $\lim S(P, f, \alpha)$  exists. Hence  $f \in \mathcal{R}(\beta_3)$  iff  $f$  is continuous at  $x = 0$ .

**Example 3.** For the functions  $\beta_1$  and  $\beta_2$  defined in Example 2, prove that  $\beta_2 \in R(\beta_1)$ , although  $\lim S(P, \beta_2, \beta_1)$  does not exist, as  $\mu(P) \rightarrow 0$ .

Let  $P = \{-1 = x_0, x_1, \dots, x_n = 1\}$  be a partition of  $[-1, 1]$  such that  $0 \in \Delta x_r$

Let  $t_i \in \Delta x_i$ , when  $i = 1, 2, 3, \dots, n$ . Now

$$\begin{aligned} S(P, \beta_2, \beta_1) &= \sum_{i=1}^n \beta_2(t_i) \{\beta_1(x_i) - \beta_1(x_{i-1})\} \\ &= \beta_2(t_r) \end{aligned}$$

$\therefore \lim_{\mu(P) \rightarrow 0} S(P, \beta_2, \beta_1) = 0$  or 1, according as  $t_r < 0$  or  $\geq 0$

Thus  $\lim S(P, \beta_2, \beta_1)$  does not exist.

Let  $P^* = P \cup \{0\}$ , and  $0 \in \Delta x_r$ .

Now

$$U(P^*, \beta_2, \beta_1) = 1 \cdot \{\beta_1(x_r) - \beta_1(0)\} = 1$$

$$L(P^*, \beta_2, \beta_1) = 1 \cdot \{\beta_1(x_r) - \beta_1(0)\} = 1$$

Thus

$$U(P^*, \beta_2, \beta_1) = L(P^*, \beta_2, \beta_1) = 1$$

$\therefore$

$$\beta_2 \in \mathcal{R}(\beta_1) \text{ and } \int_{-1}^1 \beta_2 d\beta_1 = 1$$

**Ex. 1.** For the functions  $f, \beta_1, \beta_2$  defined in Example 2, prove that

(a)  $f \in \mathcal{R}(\beta_1)$  iff  $f(0+) = f(0)$

and in that case

$$\int_{-1}^1 f d\beta_1 = f(0)$$

(b)  $f \in \mathcal{R}(\beta_2)$  iff  $f(0-) = f(0)$ ,

and in that case

$$\int_{-1}^1 f d\beta_2 = f(0)$$

**Ex. 2.** Show that

$$\int_0^4 x d([x] - x) = \frac{3}{2}$$

where  $[x]$  is the greatest integer not exceeding  $x$ .

**Ex. 3.** Show that

(i)  $\int_0^x d[t] = [x] \quad \forall x \in \mathbb{R}$

(ii)  $\int_0^4 x d[x] = 10$

(iii)  $\int_0^4 x d([x] - x) = 2$

(iv)  $\int_0^2 x^2 d(x^2) = 8$

(v)  $\int_0^2 [x] d(x^2) = 3$

(vi)  $\int_0^3 x^2 d([x] - x) = 5$

(vii)  $\int_{-1}^1 (x^2 + e^x) d(\operatorname{sgn} x) = 1$

(viii)  $\int_{\pi}^{2\pi} \sin x d(\cos x) = \frac{-\pi}{2}$

Ex 4. Let  $\alpha(x) = f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x \leq 2 \end{cases}$

and

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x \leq 2 \end{cases}$$

(i) Is  $f \in \mathcal{R}(\alpha)$ ? If so, compute  $\int_0^2 f d(\alpha)$ .

(ii) Is  $g \in \mathcal{R}(\alpha)$ ? If so, compute  $\int_0^2 g d(\alpha)$ .

Ex. 5. Evaluate

(i)  $\int_0^2 x d\alpha(x)$ , where  $\alpha(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 + x, & 1 < x \leq 2 \end{cases}$

(ii)  $\int_0^3 f(x) d([x] + x)$ , where  $f(x) = \begin{cases} [x], & 0 \leq x < 3/2 \\ e^x, & 3/2 \leq x \leq 3 \end{cases}$

## 5. SOME IMPORTANT THEOREMS

We add a few theorems before closing the discussion.

**Theorem 7.** If  $f \in \mathcal{R}[a, b]$  and  $\alpha$  is monotone increasing on  $[a, b]$  such that  $\alpha' \in \mathcal{R}[a, b]$ , then  $f \in \mathcal{R}(\alpha)$ , and

$$\int_a^b f d\alpha = \int_a^b f \alpha' dx$$

Let  $\varepsilon > 0$  be any given number.

Since  $f$  is bounded, there exists  $M > 0$ , such that

$$|f(x)| \leq M, \quad \forall x \in [a, b]$$

Again since  $f, \alpha' \in \mathcal{R}[a, b]$ , therefore  $f\alpha' \in \mathcal{R}[a, b]$  and consequently  $\exists \delta_1 > 0, \delta_2 > 0$  such that

$$\left| \sum f(t_i) \alpha'(t_i) \Delta x_i - \int f \alpha' dx \right| < \varepsilon/2 \quad (1)$$

for  $\mu(P) < \delta_1$  and all  $t_i \in \Delta x_i$ , and

$$\left| \sum \alpha'(t_i) \Delta x_i - \int \alpha' dx \right| < \varepsilon/4M \quad (2)$$

for  $\mu(P) < \delta_2$  and all  $t_i \in \Delta x_i$ .

Now for  $\mu(P) < \delta_2$  and all  $t_i \in \Delta x_i$ ,  $s_i \in \Delta x_i$ , (2) gives

$$\sum |\alpha'(t_i) - \alpha'(s_i)| \Delta x_i < 2 \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2M} \quad (3)$$

Let  $\delta = \min(\delta_1, \delta_2)$ , and  $P$  any partition with  $\mu(P) < \delta$ .

Then, for all  $t_i \in \Delta x_i$ , by Lagrange's Mean Value Theorem, there are points  $s_i \in \Delta x_i$  such that

$$\Delta \alpha_i = \alpha'(s_i) \Delta x_i \quad (4)$$

Thus

$$\begin{aligned} \left| \sum f(t_i) \Delta \alpha_i - \int f \alpha' dx \right| &= \left| \sum f(t_i) \alpha'(s_i) \Delta x_i - \int f \alpha' dx \right| \\ &= \left| \sum f(t_i) \alpha'(t_i) \Delta x_i - \int f \alpha' dx + \sum f(t_i) [\alpha'(s_i) - \alpha'(t_i)] \Delta x_i \right| \\ &\leq \left| \sum f(t_i) \alpha'(t_i) \Delta x_i - \int f \alpha' dx \right| + \sum |f(t_i)| |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i \\ &< \frac{\varepsilon}{2} + M \frac{\varepsilon}{2M} = \varepsilon \end{aligned} \quad (5)$$

Hence for any  $\varepsilon > 0$ ,  $\exists \delta > 0$  such that for all partitions with  $\mu(P) < \delta$ , (5) holds

$$\Rightarrow \lim_{\mu(P) \rightarrow 0} \sum f(t_i) \Delta \alpha_i \text{ exists and equals } \int_a^b f \alpha' dx$$

$$\Rightarrow f \in \mathcal{R}(\alpha), \text{ and } \int_a^b f d\alpha = \int_a^b f \alpha' dx$$

**Theorem 8 (A particular case).** If  $f$  is continuous on  $[a, b]$  and  $\alpha$  has a continuous derivative on  $[a, b]$ , then

$$\int_a^b f d\alpha = \int_a^b f \alpha' dx$$

Under the given conditions all the integrals exist.

Let  $P = \{a = x_0, \dots, x_n = b\}$  be any partition of  $[a, b]$ . Thus, by Lagrange's Mean Value Theorem it is possible to find  $t_i \in ]x_{i-1}, x_i[$ , such that

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i) (x_i - x_{i-1}), \quad i = 1, 2, \dots, n$$

or

$$\Delta \alpha_i = \alpha'(t_i) \Delta x_i$$

$$\therefore S(P, f, \alpha) = \sum_{i=1}^n f(t_i) \Delta \alpha_i$$

$$= \sum_{i=1}^n f(t_i) \alpha'(t_i) \Delta x_i = S(P, f \alpha') \quad (6)$$

Proceeding to limits as  $\mu(P) \rightarrow 0$ , since both the limits exist, we get

$$\int_a^b f d\alpha = \int_a^b f \alpha' dx$$

**Note 1.** The theorem illustrates one of the situations in which Reimann-Stieltjes integrals reduce to Riemann integrals.

2. In equation (6)  $\lim S(P, f, \alpha)$  exists in view of Theorem 5 while  $\lim S(P, f\alpha')$  exists because  $f\alpha'$  is continuous and hence integrable in the Riemann sense.

### Examples.

$$(i) \int_0^2 x^2 dx^2 = \int_0^2 x^2 2x dx = \int_0^2 2x^3 dx = 8$$

$$(ii) \int_0^2 [x] dx^2 = \int_0^2 [x] 2x dx \\ = \int_0^1 [x] 2x dx + \int_1^2 [x] 2x dx = 0 + 3 = 3$$

**Ex.** Evaluate the following integrals:

$$(i) \int_1^4 (x - [x]) dx^2$$

$$(ii) \int_0^3 \sqrt{x} dx^3$$

$$(iii) \int_0^3 [x] d(e^x)$$

$$(iv) \int_0^{\pi/2} x d(\sin x)$$

**Theorem 9 (First Mean Value Theorem).** If a function  $f$  is continuous on  $[a, b]$  and  $\alpha$  is monotonic increasing on  $[a, b]$ , then there exists a number  $\xi$  in  $[a, b]$  such that

$$\int_a^b f d\alpha = f(\xi) \{\alpha(b) - \alpha(a)\}$$

$f$  is continuous and  $\alpha$  is monotonic, therefore  $f \in \mathcal{R}(\alpha)$ .

Let  $m, M$  be the infimum and supremum of  $f$  in  $[a, b]$ . Then as in § 1.1,

$$m\{\alpha(b) - \alpha(a)\} \leq \int_a^b f d\alpha \leq M\{\alpha(b) - \alpha(a)\}$$

Hence there exists a number  $\mu$ ,  $m \leq \mu \leq M$  such that

$$\int_a^b f d\alpha = \mu\{\alpha(b) - \alpha(a)\}$$

Again, since  $f$  is continuous, there exists a number  $\xi \in [a, b]$  such that  $f(\xi) = \mu$

$$\therefore \int_a^b f d\alpha = f(\xi) \{\alpha(b) - \alpha(a)\}$$

**Remark.** It may not be possible always to choose  $\xi$  such that  $a < \xi < b$ .

$$\text{Consider } \alpha(x) = \begin{cases} 0, & x = a \\ 1, & a < x \leq b \end{cases}$$

For a continuous function  $f$ , we have

$$\int_a^b f d\alpha = f(a) = f(a) \{\alpha(b) - \alpha(a)\}$$

**Theorem 10.** If  $f$  is continuous and  $\alpha$  monotone on  $[a, b]$ , then

$$\int_a^b f d\alpha = [f(x)\alpha(x)]_a^b - \int_a^b \alpha df$$

Under the given conditions all the integrals exist by Theorem 5.

Let  $P = \{a = x_0, x_1, \dots, x_n = b\}$  be a partition of  $[a, b]$ .

Choose  $t_1, t_2, \dots, t_n$  such that  $x_{i-1} \leq t_i \leq x_i$ , and let  $t_0 = a, t_{n+1} = b$ , so that  $t_{i-1} \leq x_{i-1} \leq t_i$ .

Clearly  $Q = \{a = t_0, t_1, t_2, \dots, t_n, t_{n+1} = b\}$  is also a partition of  $[a, b]$ .

Now

$$\begin{aligned} S(P, f, \alpha) &= \sum_{i=1}^n f(t_i) \Delta \alpha_i \\ &= f(t_1)[\alpha(x_1) - \alpha(x_0)] + f(t_2)[\alpha(x_2) - \alpha(x_1)] + \dots \\ &\quad + f(t_n)[\alpha(x_n) - \alpha(x_{n-1})] \\ &= -\alpha(x_0)f(t_1) - \alpha(x_1)[f(t_2) - f(t_1)] + \alpha(x_2)[f(t_3) - f(t_2)] + \dots \\ &\quad + \alpha(x_{n-1})[f(t_n) - f(t_{n-1})] + \alpha(x_n)f(t_n) \end{aligned}$$

Adding and subtracting  $\alpha(x_0)f(t_0) + \alpha(x_n)f(t_{n+1})$ , we get

$$\begin{aligned} S(P, f, \alpha) &= \alpha(x_n)f(t_{n+1}) - \alpha(x_0)f(t_0) - \sum_{i=0}^n \alpha(x_i)\{f(t_{i+1}) - f(t_i)\} \\ &= f(b)\alpha(b) - f(a)\alpha(a) - S(Q, \alpha, f) \end{aligned} \quad (1)$$

If  $\mu(P) \rightarrow 0$ , then  $\mu(Q) \rightarrow 0$  and Theorem 5 shows that  $\lim S(P, f, \alpha)$  and  $\lim S(Q, \alpha, f)$  both exist and that

$$\lim S(P, f, \alpha) = \int_a^b f d\alpha$$

and

$$\lim S(Q, \alpha, f) = \int_a^b \alpha df$$

Hence proceeding to limits when  $\mu(P) \rightarrow 0$ , we get from (1),

$$\int_a^b f d\alpha = [f(x) \alpha(x)]_a^b - \int_a^b \alpha df \quad (2)$$

where  $[f(x) \alpha(x)]_a^b$  denotes the difference  $f(b)\alpha(b) - f(a)\alpha(a)$ .

**Remark.** The theorem holds when one of the functions is continuous and the other monotone.

**Note.** The theorem is similar to the theorem, 'Integration by parts' for Riemann integration.

**Corollary.** The result of the theorem can be put in a slightly different form, by using Theorem 9, if, in addition to monotonicity  $\alpha$  is continuous also

$$\begin{aligned} \int_a^b f d\alpha &= f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha df \\ &= f(b)\alpha(b) - f(a)\alpha(a) - a(\xi) [f(b) - f(a)] \\ &= f(b)[\alpha(\xi) - \alpha(a)] + f(b) [\alpha(b) - \alpha(\xi)] \end{aligned}$$

where  $\xi \in [a, b]$ .

Stated in this form, it is called the **Second Mean Value Theorem**.

**Theorem 11 (Change of variable).** If

(i)  $f$  is a continuous function on  $[a, b]$ , and

(ii)  $\phi$  is a continuous and strictly monotonic function on  $[\alpha, \beta]$  where  $a = \phi(\alpha)$ ,  $b = \phi(\beta)$

then

$$\int_a^b f(x) dx = \int_{\alpha}^{\beta} f(\phi(y)) d\phi(y)$$

$$\left[ \text{Change of variable in } \int_a^b f(x) dx \text{ by putting } x = \phi(y) \right]$$

Let  $\phi$  be strictly monotonic increasing.

Since  $\phi$  is strictly monotonic, it is invertible, i.e.,

$$x = \phi(y) \Rightarrow y = \phi^{-1}(x), \quad \forall x \in [a, b]$$

so that

$$\alpha = \phi^{-1}(a), \quad \beta = \phi^{-1}(b)$$

Let

$$P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$$

be any partition of  $[a, b]$ , and

$$Q = (\alpha = y_0, y_1, y_2, \dots, y_n = \beta), \quad y_i = \phi^{-1}(x_i)$$

be the corresponding partition of  $[\alpha, \beta]$ , so that

$$\Delta x_i = x_i - x_{i-1} = \phi(y_i) - \phi(y_{i-1}) = \Delta \phi_i \quad (1)$$

Again, for any  $\xi_i \in \Delta x_i$ , let  $\eta_i \in \Delta y_i$  where

$$\xi_i = \phi(\eta_i) \quad (2)$$

Putting  $g(y) = f(\phi(y))$ , we have

$$\begin{aligned} S(P, f) &= \sum_i f(\xi_i) \Delta x_i \\ &= \sum_i f(\phi(\eta_i)) \Delta \phi_i = \sum_i g(\eta_i) \Delta \phi_i \\ &= S(Q, g, \phi) \end{aligned} \quad (3)$$

Continuity of  $f$  implies that  $S(P, f) \rightarrow \int_a^b f dx$  as  $\mu(P) \rightarrow 0$ . Also continuity of  $g$  implies (by

Theorem 5) that  $S(Q, g, \phi) \rightarrow \int_\alpha^\beta g(y) d\phi$  as  $\mu(P) \rightarrow 0$ .

Since uniform continuity of  $\phi$  on  $[\alpha, \beta]$  implies that  $\mu(Q) \rightarrow 0$  as  $\mu(P) \rightarrow 0$ , therefore letting  $\mu(P) \rightarrow 0$  in (3), we get

$$\int_a^b f(x) dx = \int_\alpha^\beta g(y) d\phi = \int_\alpha^\beta f(\phi(y)) d\phi(y).$$