Since a geometric series with a ratio less than 1 converges, f is properly defined. Moreover,

$$\text{if } a < u < v < b, \text{ then } f(v) - f(u) = \sum_{\{n \mid u < q_n \leq v\}} \frac{1}{2^n}. \tag{1}$$

Thus f is increasing. Let $x_0 = q_k$ belong to C. Then, by (1),

$$f(x_0) - f(x) \geq \frac{1}{2^k} \text{ for all } x < x_0.$$

Therefore f fails to be continuous at x_0. Now let x_0 belong to $(a, b) \sim C$. Let n be a natural number. There is an open interval I containing x_0 for which q_n does not belong to I for $1 \leq k \leq n$. We infer from (1) that $|f(x) - f(x_0)| < 1/2^n$ for all $x \in I$. Therefore f is continuous at x_0. □

PROBLEMS

1. Let C be a countable subset of the nondegenerate closed, bounded interval $[a, b]$. Show that there is an increasing function on $[a, b]$ that is continuous only at points in $[a, b] \sim C$.
2. Show that there is a strictly increasing function on $[0, 1]$ that is continuous only at the irrational numbers in $[0, 1]$.
3. Let f be a monotone function on a subset E of $\mathbf{R}$. Show that f is continuous except possibly at a countable number of points in E.
4. Let E be a subset of $\mathbf{R}$ and C a countable subset of E. Is there a monotone function on E that is continuous only at points in $E \sim C$?

6.2 DIFFERENTIABILITY OF MONOTONE FUNCTIONS: LEBESGUE'S THEOREM

A closed, bounded interval $[c, d]$ is said to be nondegenerate provided $c < d$.

Definition *A collection $\mathcal{F}$ of closed, bounded, nondegenerate intervals is said to cover a set E* **in the sense of Vitali** *provided for each point x in E and $\epsilon > 0$, there is an interval I in $\mathcal{F}$ that contains x and has $\ell(I) < \epsilon$.*

The Vitali Covering Lemma *Let E be a set of finite outer measure and $\mathcal{F}$ a collection of closed, bounded intervals that covers E in the sense of Vitali. Then for each $\epsilon > 0$, there is a finite disjoint subcollection $\{I_k\}_{k=1}^n$ of $\mathcal{F}$ for which*

$$m^*\left[E \sim \bigcup_{k=1}^n I_k\right] < \epsilon. \tag{2}$$

Proof Since $m^*(E) < \infty$, there is an open set $\mathcal{O}$ containing E for which $m(\mathcal{O}) < \infty$. Because $\mathcal{F}$ is a Vitali covering of E, we may assume that each interval in $\mathcal{F}$ is contained in $\mathcal{O}$. By the countable additivity and monotonicity of measure,

$$\text{if } \{I_k\}_{k=1}^\infty \subseteq \mathcal{F} \text{ is disjoint, then } \sum_{k=1}^\infty \ell(I_k) \leq m(\mathcal{O}) < \infty. \tag{3}$$

Moreover, since each I_k is closed and $\mathcal{F}$ is a Vitali covering of E,

$$\text{if } \{I_k\}_{k=1}^n \subseteq \mathcal{F}\text{, then } E \sim \bigcup_{k=1}^{\infty} I_k \subseteq \bigcup_{I \in \mathcal{F}_n} I \text{ where } \mathcal{F}_n = \left\{ I \in \mathcal{F} \;\middle|\; I \cap \bigcup_{k=1}^{n} I_k = \emptyset \right\}. \tag{4}$$

If there is a finite disjoint subcollection of $\mathcal{F}$ that covers E, the proof is complete. Otherwise, we inductively choose a disjoint countable subcollection $\{I_k\}_{k=1}^{\infty}$ of $\mathcal{F}$ which has the following property:

$$E \sim \bigcup_{k=1}^{n} I_k \subseteq \bigcup_{k=n+1}^{\infty} 5 * I_k \text{ for all } n, \tag{5}$$

where, for a closed, bounded interval I, $5 * I$ denotes the closed interval that has the same midpoint as I and 5 times its length. To begin this selection, let I_1 be any interval in $\mathcal{F}$. Suppose n is a natural number and the finite disjoint subcollection $\{I_k\}_{k=1}^{n}$ of $\mathcal{F}$ has been chosen. Since $E \sim \bigcup_{k=1}^{n} I_k \neq \emptyset$, the collection $\mathcal{F}_n$ defined in (4) is nonempty. Moreover, the supremum, s_n, of the lengths of the intervals in $\mathcal{F}_n$ is finite since $m(\mathcal{O})$ is an upper bound for these lengths. Choose I_{n+1} to be an interval in $\mathcal{F}_n$ for which $\ell(I_{n+1}) > s_n/2$. This inductively defines $\{I_k\}_{k=1}^{\infty}$, a countable disjoint subcollection of $\mathcal{F}$ such that for each n,

$$\ell(I_{n+1}) > \ell(I)/2 \text{ if } I \in \mathcal{F} \text{ and } I \cap \bigcup_{k=1}^{n} I_k = \emptyset. \tag{6}$$

We infer from (3) that $\{\ell(I_k)\} \to 0$. Fix a natural number n. To verify the inclusion (5), let x belong to $E \sim \bigcup_{k=1}^{n} I_k$. We infer from (4) that there is an $I \in \mathcal{F}$ which contains x and is disjoint from $\bigcup_{k=1}^{n} I_k$. Now I must have nonempty intersection with some I_k, for otherwise, by (6), $\ell(I_k) > \ell(I)/2$ for all k, which contradicts the convergence of $\{\ell(I_k)\}$ to 0. Let N be the first natural number for which $I \cap I_N \neq \emptyset$. Then $N > n$. Since $I \cap \bigcup_{k=1}^{N-1} I_k = \emptyset$, we infer from (6) that $\ell(I_N) > \ell(I)/2$. Since x belongs to I and $I \cap I_N \neq \emptyset$, the distance from x to the midpoint of I_N is at most $\ell(I) + 1/2 \cdot \ell(I_N)$ and hence, since $\ell(I) < 2 \cdot \ell(I_N)$, the distance from x to the midpoint of I_N is less than $5/2 \cdot \ell(I_N)$. This means that x belongs to $5 * I_N$. Thus,

$$x \in 5 * I_N \subseteq \bigcup_{k=n+1}^{\infty} 5 * I_k.$$

We have established the inclusion (5).

Let $\epsilon > 0$. We infer from (3) that here is a natural number n for which $\sum_{k=n+1}^{\infty} \ell(I_k) < \epsilon/5$. This choice of n, together with the inclusion (5) and the monotonicity and countable additivity of measure, establishes (2).

□

For a real-valued function f and an interior point x of its domain, the **upper derivative** of f at x, $\overline{D}f(x)$ and the **lower derivative** of f at x, $\underline{D}f(x)$ are defined as follows:

$$\overline{D}f(x) = \lim_{h \to 0} \left[\sup_{0 < |t| \leq h} \frac{f(x+t) - f(x)}{t} \right];$$

$$\underline{D}f(x) = \lim_{h \to 0} \left[\inf_{0 < |t| \leq h} \frac{f(x+t) - f(x)}{t} \right].$$

We have $\overline{D}f(x) \geq \underline{D}f(x)$. If $\overline{D}f(x)$ equals $\underline{D}f(x)$ and is finite, we say that f is **differentiable** at x and define $f'(x)$ to be the common value of the upper and lower derivatives.

The Mean Value Theorem of calculus tells us that if a function f is continuous on the closed, bounded interval $[c, d]$ and differentiable on its interior (c, d) with $f' \geq \alpha$ on (c, d), then

$$\alpha \cdot (d-c) \leq [f(d) - f(c)].$$

The proof of the following generalization of this inequality, inequality (7), is a nice illustration of the fruitful interplay between the Vitali Covering Lemma and monotonicity properties of functions.

Lemma 3 *Let f be an increasing function on the closed, bounded interval $[a, b]$. Then, for each $\alpha > 0$,*

$$m^*\{x \in (a, b) \mid \overline{D}f(x) \geq \alpha\} \leq \frac{1}{\alpha} \cdot [f(b) - f(a)] \tag{7}$$

and

$$m^*\{x \in (a, b) \mid \overline{D}f(x) = \infty\} = 0. \tag{8}$$

Proof Let $\alpha > 0$. Define $E_\alpha = \{x \in (a, b) \mid \overline{D}f(x) \geq \alpha\}$. Choose $\alpha' \in (0, \alpha)$. Let $\mathcal{F}$ be the collection of closed, bounded intervals $[c, d]$ contained in (a, b) for which $f(d) - f(c) \geq \alpha'(d-c)$. Since $\overline{D}f \geq \alpha$ on E_α, $\mathcal{F}$ is a Vitali covering of E_α. The Vitali Covering Lemma tells us that there is a finite disjoint subcollection $\{[c_k, d_k]\}_{k=1}^n$ of $\mathcal{F}$ for which

$$m^*\left[E_\alpha \sim \bigcup_{k=1}^{n} [c_k, d_k]\right] < \epsilon.$$

Since $E_\alpha \subseteq \bigcup_{k=1}^n [c_k, d_k] \cup \{E_\alpha \sim \bigcup_{k=1}^n [c_k, d_k]\}$, by the finite subadditivity of outer measure, the preceding inequality and the choice of the intervals $[c_k, d_k]$,

$$m^*(E_\alpha) < \sum_{k=1}^{n} (d_k - c_k) + \epsilon \leq \frac{1}{\alpha'} \cdot \sum_{k=1}^{n} [f(d_k) - f(c_k)] + \epsilon. \tag{9}$$

However, the function f is increasing on $[a, b]$ and $\{[c_k, d_k]\}_{k=1}^n$ is a disjoint collection of subintervals of $[a, b]$. Therefore

$$\sum_{k=1}^{n} [f(d_k) - f(c_k)] \leq f(b) - f(a).$$

Thus for each $\epsilon > 0$, and each $\alpha' \in (0, \alpha)$,

$$m^*(E_\alpha) \leq \frac{1}{\alpha'} \cdot [f(b) - f(a)] + \epsilon.$$

This proves (7). For each natural number n, $\{x \in (a, b) \mid \overline{D}f(x) = \infty\} \subseteq E_n$ and therefore

$$m^*\{x \in (a, b) \mid \overline{D}f(x) = \infty\} \leq m^*(E_n) \leq \frac{1}{n} \cdot (f(b) - f(a)).$$

This proves (8). $\square$

Lebesgue's Theorem *If the function f is monotone on the open interval (a, b), then it is differentiable almost everywhere on (a, b).*

Proof Assume f is increasing. Furthermore, assume (a, b) is bounded. Otherwise, express (a, b) as the union of an ascending sequence of open, bounded intervals and use the continuity of Lebesgue measure. The set of points x in (a, b) at which $\overline{D}f(x) > \underline{D}f(x)$ is the union of the sets

$$E_{\alpha,\beta} = \{x \in (a, b) \mid \overline{D}f(x) > \alpha > \beta > \underline{D}f(x)\}$$

where α and β are rational numbers. Hence, since this is a countable collection, by the countable subadditivity of outer measure, it suffices to prove that each $E_{\alpha,\beta}$ has outer measure zero. Fix rationals α, β with $\alpha > \beta$ and set $E = E_{\alpha,\beta}$. Let $\epsilon > 0$. Choose an open set $\mathcal{O}$ for which

$$E \subseteq \mathcal{O} \subseteq (a, b) \text{ and } m(\mathcal{O}) < m^*(E) + \epsilon. \tag{10}$$

Let $\mathcal{F}$ be the collection of closed, bounded intervals $[c, d]$ contained in $\mathcal{O}$ for which $f(d) - f(c) < \beta(d - c)$. Since $\underline{D}f < \beta$ on E, $\mathcal{F}$ is a Vitali covering of E. The Vitali Covering Lemma tells us that there is a finite disjoint subcollection $\{[c_k, d_k]\}_{k=1}^n$ of $\mathcal{F}$ for which

$$m^*\left[E \sim \bigcup_{k=1}^{n} [c_k, d_k]\right] < \epsilon. \tag{11}$$

By the choice of the intervals $[c_k, d_k]$, the inclusion of the union of the disjoint collection intervals $\{[c_k, d_k]\}_{k=1}^n$ in $\mathcal{O}$ and (10),

$$\sum_{k=1}^{n} [f(d_k) - f(c_k)] < \beta\left[\sum_{k=1}^{n} (d_k - c_k)\right] \leq \beta \cdot m(\mathcal{O}) \leq \beta \cdot [m^*(E) + \epsilon]. \tag{12}$$

For $1 \leq k \leq n$, we infer from the preceding lemma, applied to the restriction of f to $[c_k, d_k]$, that

$$m^*(E \cap (c_k, d_k)) \leq \frac{1}{\alpha}[f(d_k) - f(c_k)].$$

Therefore, by (11),

$$m^*(E) \leq \sum_{k=1}^{n} m^*(E \cap (c_k, d_k)) + \epsilon \leq \frac{1}{\alpha}\left[\sum_{k=1}^{n} [f(d_k) - f(c_k)]\right] + \epsilon. \tag{13}$$

We infer from (12) and (13) that

$$m^*(E) \leq \frac{\beta}{\alpha} \cdot m^*(E) + \frac{1}{\alpha} \cdot \epsilon + \epsilon \text{ for all } \epsilon > 0.$$

Therefore, since $0 \leq m^*(E) < \infty$ and $\beta/\alpha < 1$, $m^*(E) = 0$. □

Lebesgue's Theorem is the best possible in the sense that if E is a set of measure zero contained in the open interval (a, b), there is an increasing function on (a, b) that fails to be differentiable at each point in E (see Problem 10).

Remark *Frigyes Riesz and Béla Sz.-Nagy*[2] *remark that Lebesgue's Theorem is "one of the most striking and most important in real variable theory." Indeed, in 1872 Karl Weierstrass presented mathematics with a continuous function on an open interval which failed to be differentiable at any point.*[3] *Further pathology was revealed and there followed a period of uncertainty regarding the spread of pathology in mathematical analysis. Lebesgue's Theorem, which was published in 1904, and its consequences, which we pursue in Section 5, helped restore confidence in the harmony of mathematics analysis.*

Let f be integrable over the closed, bounded interval $[a, b]$. Extend f to take the value $f(b)$ on $(b, b+1]$. For $0 < h \leq 1$, define the **divided difference function** $\text{Diff}_h\, f$ and **average value function** $\text{Av}_h\, f$ of $[a, b]$ by

$$\text{Diff}_h\, f(x) = \frac{f(x+h) - f(x)}{h} \text{ and } \text{Av}_h\, f(x) = \frac{1}{h} \cdot \int_x^{x+h} f \text{ for all } x \in [a, b].$$

By a change of variables in the integral and cancellation, for all $a \leq u < v \leq b$,

$$\int_u^v \text{Diff}_h\, f = \text{Av}_h\, f(v) - \text{Av}_h\, f(u). \tag{14}$$

Corollary 4 *Let f be an increasing function on the closed, bounded interval $[a, b]$. Then f' is integrable over $[a, b]$ and*

$$\int_b^a f' \leq f(b) - f(a). \tag{15}$$

Proof Since f is increasing on $[a, b+1]$, it is measurable (see Problem 22) and therefore the divided difference functions are also measurable. Lebesgue's Theorem tells us that f is differentiable almost everywhere on (a, b). Therefore $\{\text{Diff}_{1/n}\, f\}$ is a sequence of nonnegative measurable functions that converges pointwise almost everywhere on $[a, b]$ to f'. According to Fatou's Lemma,

$$\int_a^b f' \leq \liminf_{n \to \infty} \left[\int_a^b \text{Diff}_{1/n}\, f \right] \tag{16}$$

By the change of variable formula (14), for each natural number n, since f is increasing,

$$\int_a^b \text{Diff}_{1/n}\, f = \frac{1}{1/n} \cdot \int_b^{b+1/n} f - \frac{1}{1/n} \cdot \int_a^{a+1/n} f = f(b) - \frac{1}{1/n} \cdot \int_a^{a+1/n} \leq f(b) - f(a).$$

Thus

$$\limsup_{n \to \infty} \left[\int_a^b \text{Diff}_{1/n}\, f \right] \leq f(b) - f(a). \tag{17}$$

The inequality (15) follows from the inequalities (16) and (17). □

[2] See page 5 of their book *Functional Analysis* [RSN90].

[3] A simpler example of such a function, due to Bartel van der Waerden, is examined in Chapter 8 of Patrick Fitzpatrick's *Advanced Calculus* [Fit09].

Remark *The integral in (15) is independent of the values taken by f at the endpoints. On the other hand, the right-hand side of this equality holds for the extension of any increasing extension of f on the open, bounded interval (a, b) to its closure $[a, b]$. Therefore a tighter form of equality (15) is*

$$\int_a^b f' \leq \sup_{x\in(a,b)} f(x) - \inf_{x\in(a,b)} f(x). \tag{18}$$

The right-hand side of this inequality equals $f(b)-f(a)$ if and only if f is continuous at the endpoints. However, even if f is increasing and continuous on $[a, b]$, inequality (15) may be strict. It is strict for the Cantor-Lebesgue function φ on $[0, 1]$ since $\varphi(1)-\varphi(0)=1$ while φ' vanishes almost everywhere on $(0, 1)$. We show that for an increasing function f on $[a, b]$, (15) is an equality if and only if the function is absolutely continuous on $[a, b]$ (see the forthcoming Corollary 12).

Remark *For a continuous function f on a closed, bounded interval $[a, b]$ that is differentiable on the open interval (a, b), in the absence of a monotonicity assumption on f we cannot infer that its derivative f' is integrable over $[a, b]$. We leave it as an exercise to show that for f defined on $[0, 1]$ by*

$$f(x) = \begin{cases} x^2\sin(1/x^2) & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0, \end{cases}$$

f' is not integrable over $[0, 1]$.

PROBLEMS

5. Show that the Vitali Covering Lemma does not extend to the case in which the covering collection has degenerate closed intervals.
6. Show that the Vitali Covering Lemma does extend to the case in which the covering collection consists of nondegenerate general intervals.
7. Let f be continuous on $\mathbf{R}$. Is there an open interval on which f is monotone?
8. Let I and J be closed, bounded intervals and $\gamma > 0$ be such that $\ell(I) > \gamma \cdot \ell(J)$. Assume $I \cap J \neq \emptyset$. Show that if $\gamma \geq 1/2$, then $J \subseteq 5 * I$, where $5 * I$ denotes the interval with the same center as I and five times its length. Is the same true if $0 < \gamma < 1/2$?
9. Show that a set E of real numbers has measure zero if and only if there is a countable collection of open intervals $\{I_k\}_{k=1}^\infty$ for which each point in E belongs to infinitely many of the I_k's and $\sum_{k=1}^\infty \ell(I_k) < \infty$.
10. (Riesz-Nagy) Let E be a set of measure zero contained in the open interval (a, b). According to the preceding problem, there is a countable collection of open intervals contained in (a, b), $\{(c_k, d_k)\}_{k=1}^\infty$, for which each point in E belongs to infinitely many intervals in the collection and $\sum_{k=1}^\infty (d_k - c_k) < \infty$. Define

$$f(x) = \sum_{k=1}^\infty \ell((c_k, d_k) \cap (-\infty, x)) \text{ for all } x \text{ in } (a, b).$$

Show that f is increasing and fails to be differentiable at each point in E.

23. Show that a continuous function f on $[a, b]$ is Lipschitz if its upper and lower derivatives are bounded on (a, b).
24. Show that for f defined in the last remark of this section, f' is not integrable over $[0, 1]$.

6.3 FUNCTIONS OF BOUNDED VARIATION: JORDAN'S THEOREM

Lebesgue's Theorem tells us that a monotone function on an open interval is differentiable almost everywhere. Therefcre the *difference* of two increasing functions on an open interval also is differentiable almost everywhere. We now provide a characterization of the class of functions on a closed, bounded interval that may be expressed as the difference of increasing functions, which shows that this class is surprisingly large: it includes, for instance, all Lipschitz functions.

Let f be a real-valued function defined on the closed, bounded interval $[a, b]$ and $P = \{x_0, \ldots, x_k\}$ be a partition of $[a, b]$. Define the variation of f with respect to P by

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|,$$

and the **total variation** of f on $[a, b]$ by

$$TV(f) = \sup\left\{V(f, P) \mid P \text{ a partition of } [a, b]\right\}.$$

For a subinterval $[c, d]$ of $[a, b]$, $TV(f_{[c,d]})$ denotes the total variation of the restriction of f to $[c, d]$.

Definition *A real-valued function f on the closed, bounded interval $[a, b]$ is said to be of* **bounded variation** *on $[a, b]$ provided*

$$TV(f) < \infty.$$

Example Let f be an increasing function on $[a, b]$. Then f is of bounded variation on $[a, b]$ and

$$TV(f) = f(b) - f(a).$$

Indeed, for any partition $P = \{x_0, \ldots, x_k\}$ of $[a, b]$,

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})] = f(b) - f(a).$$

Example Let f be a Lipschitz function on $[a, b]$. Then f is of bounded variation of $[a, b]$, and $TV(f) \leq c \cdot (b - a)$, where

$$|f(u) - f(v)| \leq c|u - v| \text{ for all } u, v \text{ in } [a, b].$$

Indeed, for a partition $P = \{x_0, \ldots, x_k\}$ of $[a, b]$,

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \leq c \cdot \sum_{i=1}^{k} [x_i - x_{i-1}] = c \cdot [b - a].$$

Thus, $c \cdot [b-a]$ is an upper bound of the set of all variations of f with respect to a partition of $[a, b]$ and hence $TV(f) \leq c \cdot [b-a]$.

Example Define the function f on $[0, 1]$ by

$$f(x) = \begin{cases} x \cos(\pi/2x) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is continuous on $[0, 1]$. But f is not of bounded variation on $[0, 1]$. Indeed, for a natural number n, consider the partition $P_n = \{0, 1/2n, 1/[2n-1], \ldots, 1/3, 1/2, 1\}$ of $[0, 1]$. Then

$$V(f, P_n) = 1 + 1/2 + \ldots + 1/n.$$

Hence f is not of bounded variation on $[0, 1]$, since the harmonic series diverges.

Observe that if c belongs to (a, b), P is a partition of $[a, b]$, and P' is the refinement of P obtained by adjoining c to P, then, by the triangle inequality, $V(f, P) \leq V(f, P')$. Thus, in the definition of the total variation of a function on $[a, b]$, the supremum can be taken over partitions of $[a, b]$ that contain the point c. Now a partition P of $[a, b]$ that contains the point c induces, and is induced by, partitions P_1 and P_2 of $[a, c]$ and $[c, b]$, respectively, and for such partitions

$$V(f_{[a,b]}, P) = V(f_{[a,c]}, P_1) + V(f_{[c,b]}, P_2). \tag{19}$$

Take the supremum among such partitions to conclude that

$$TV(f_{[a,b]}) = TV(f_{[a,c]}) + TV(f_{[c,b]}). \tag{20}$$

We infer from this that if f is of bounded variation on $[a, b]$, then

$$TV(f_{[a,v]}) - TV(f_{[a,u]}) = TV(f_{[u,v]}) \geq 0 \text{ for all } a \leq u < v \leq b. \tag{21}$$

Therefore the function $x \mapsto TV(f_{[a,x]})$, which we call the **total variation function** for f, is a real-valued increasing function on $[a, b]$. Moreover, for $a \leq u < v \leq b$, if we take the crudest partition $P = \{u, v\}$ of $[u, v]$, we have

$$f(u) - f(v) \leq |f(v) - f(u)| = V(f_{[u,v]}, P) \leq TV(f_{[u,v]}) = TV(f_{[a,v]}) - TV(f_{[a,u]}).$$

Thus

$$f(v) + TV(f_{[a,v]}) \geq f(u) + TV(f_{[a,u]}) \text{ for all } a \leq u < v \leq b. \tag{22}$$

We have established the following lemma.

Lemma 5 *Let the function f be of bounded variation on the closed, bounded interval $[a, b]$. Then f has the following explicit expression as the difference of two increasing functions on $[a, b]$:*

$$f(x) = [f(x) + TV(f_{[a,x]})] - TV(f_{[a,x]}) \text{ for all } x \in [a, b]. \tag{23}$$

Jordan's Theorem *A function f is of bounded variation on the closed, bounded interval $[a, b]$ if and only if it is the difference of two increasing functions on $[a, b]$.*

Proof Let f be of bounded variation on $[a, b]$. The preceding lemma provides an explicit representation of f as the difference of increasing functions. To prove the converse, let $f = g - h$ on $[a, b]$, where g and h are increasing functions on $[a, b]$. For any partition $P = \{x_0, \ldots, x_k\}$ of $[a, b]$,

$$\begin{aligned}
V(f, P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\
&= \sum_{i=1}^{k} |[g(x_i) - g(x_{i-1})] + [h(x_{i-1}) - h(x_i)]| \\
&\leq \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| + \sum_{i=1}^{k} |h(x_{i-1}) - h(x_i)| \\
&= \sum_{i=1}^{k} [g(x_i) - g(x_{i-1})] + \sum_{i=1}^{k} [h(x_i) - h(x_{i-1})] \\
&= [g(b) - g(a)] + [h(b) - h(a)].
\end{aligned}$$

Thus, the set of variations of f with respect to partitions of $[a, b]$ is bounded above by $[g(b) - g(a)] + [h(b) - h(a)]$ and therefore f is of bounded variation of $[a, b]$. □

We call the expression of a function of bounded variation f as the difference of increasing functions a **Jordan decomposition** of f.

Corollary 6 *If the function f is of bounded variation on the closed, bounded interval $[a, b]$, then it is differentiable almost everywhere on the open interval (a, b) and f' is integrable over $[a, b]$.*

Proof According to Jordan's Theorem, f is the difference of two increasing functions on $[a, b]$. Thus Lebesgue's Theorem tells us that f is the difference of two functions which are differentiable almost everywhere on (a, b). Therefore f is differentiable almost everywhere on (a, b). The integrability of f' follows from Corollary 4. □

PROBLEMS

25. Suppose f is continuous on $[0, 1]$. Must there be a nondegenerate closed subinterval $[a, b]$ of $[0, 1]$ for which the restriction of f to $[a, b]$ is of bounded variation?
26. Let f be the Dirichlet function, the characteristic function of the rationals in $[0, 1]$. Is f of bounded variation on $[0, 1]$?
27. Define $f(x) = \sin x$ on $[0, 2\pi]$. Find two increasing functions h and g for which $f = h - g$ on $[0, 2\pi]$.
28. Let f be a step function on $[a, b]$. Find a formula for its total variation.
29. (a) Define

$$f(x) = \begin{cases} x^2 \cos(1/x^2) & \text{if } x \neq 0, x \in [-1, 1] \\ 0 & \text{if } x = 0. \end{cases}$$

Is f of bounded variation on $[-1, 1]$?

(b) Define

$$g(x) = \begin{cases} x^2 \cos(1/x) & \text{if } x \neq 0, x \in [-1, 1] \\ 0 & \text{if } x = 0. \end{cases}$$

Is g of bounded variation on $[-1, 1]$?

30. Show that the linear combination of two functions of bounded variation is also of bounded variation. Is the product of two such functions also of bounded variation?

31. Let P be a partition of $[a, b]$ that is a refinement of the partition P'. For a real-valued function f on $[a, b]$, show that $V(f, P') \leq V(f, P)$.

32. Assume f is of bounded variation on $[a, b]$. Show that there is a sequence of partitions $\{P_n\}$ of $[a, b]$ for which the sequence $\{TV(f, P_n)\}$ is increasing and converges to $TV(f)$.

33. Let $\{f_n\}$ be a sequence of real-valued functions on $[a, b]$ that converges pointwise on $[a, b]$ to the real-valued function f. Show that

$$TV(f) \leq \liminf TV(f_n).$$

34. Let f and g be of bounded variation on $[a, b]$. Show that

$$TV(f+g) \leq TV(f) + TV(g) \text{ and } TV(\alpha f) = |\alpha| TV(f).$$

35. For α and β positive numbers, define the function f on $[0, 1]$ by

$$f(x) = \begin{cases} x^\alpha \sin(1/x^\beta) & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0. \end{cases}$$

Show that if $\alpha > \beta$, then f is of bounded variation on $[0, 1]$, by showing that f' is integrable over $[0, 1]$. Then show that if $\alpha \leq \beta$, then f is not of bounded variation on $[0, 1]$.

36. Let f fail to be of bounded variation on $[0, 1]$. Show that there is a point x_0 in $[0, 1]$ such that f fails to be of bounded variation on each nondegenerate closed subinterval of $[0, 1]$ that contains x_0.

6.4 ABSOLUTELY CONTINUOUS FUNCTIONS

Definition *A real-valued function f on a closed, bounded interval $[a, b]$ is said to be **absolutely continuous** on $[a, b]$ provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b),*

$$\textit{if } \sum_{k=1}^{n} [b_k - a_k] < \delta, \textit{ then } \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon.$$

The criterion for absolute continuity in the case the finite collection of intervals consists of a single interval is the criterion for the uniform continuity of f on $[a, b]$. Thus absolutely continuous functions are continuous. The converse is false, even for increasing functions.

Example The Cantor-Lebesgue function φ is increasing and continuous on $[0, 1]$, but it is not absolutely continuous (see also Problems 40 and 48). Indeed, to see that φ is not absolutely continuous, let n be a natural number. At the n-th stage of the construction of the Cantor set, a disjoint collection $\{[c_k, d_k]\}_{1\leq k\leq 2^n}$ of 2^n subintervals of $[0, 1]$ have been constructed that cover the Cantor set, each of which has length $(1/3)^n$. The Cantor-Lebesgue function is constant on each of the intervals that comprise the complement in $[0, 1]$ of this collection of intervals. Therefore, since φ is increasing and $\varphi(1)-\varphi(0)=1$,

$$\sum_{1\leq k\leq 2^n}[d_k-c_k]=(2/3)^n \text{ while } \sum_{1\leq k\leq 2^n}[\varphi(d_k)-\varphi(c_k)]=1.$$

There is no response to the $\epsilon = 1$ challenge regarding the criterion for φ to be absolutely continuous.

Clearly linear combinations of absolutely continuous functions are absolutely continuous. However, the composition of absolutely continuous functions may fail to be absolutely continuous (see Problems 43, 44, and 45).

Proposition 7 *If the function f is Lipschitz on a closed, bounded interval $[a, b]$, then it is absolutely continuous on $[a, b]$.*

Proof Let $c > 0$ be a Lipschitz constant for f on $[a, b]$, that is,

$$|f(u)-f(v)|\leq c\,|u-v| \text{ for all } u, v\in[a, b].$$

Then, regarding the criterion for the absolute continuity of f, it is clear that $\delta=\epsilon/c$ responds to any $\epsilon>0$ challenge.

□

There are absolutely continuous functions that fail to be Lipschitz: the function f on $[0, 1]$, defined by $f(x)=\sqrt{x}$ for $0\leq x\leq 1$, is absolutely continuous but not Lipschitz (see Problem 37).

Theorem 8 *Let the function f be absolutely continuous on the closed, bounded interval $[a, b]$. Then f is the difference of increasing absolutely continuous functions and, in particular, is of bounded variation.*

Proof We first prove that f is of bounded variation. Indeed, let δ respond to the $\epsilon = 1$ challenge regarding the criterion for the absolute continuity of f. Let P be a partition of $[a, b]$ into N closed intervals $\{[c_k, d_k]\}_{k=1}^N$, each of length less than δ. Then, by the definition of δ in relation to the absolute continuity of f, it is clear that $TV(f_{[c_k, d_k]})\leq 1$, for $1\leq k\leq n$. The additivity formula (19) extends to finite sums. Hence

$$TV(f)=\sum_{k=1}^{N}TV(f_{[c_k, d_k]})\leq N.$$

Therefore f is of bounded variation. In view of (23) and the absolute continuity of sums of absolutely continuous functions, to show that f is the difference of increasing absolutely continuous functions it suffices to show that the total variation function for f is absolutely continuous. Let $\epsilon > 0$. Choose δ as a response to the $\epsilon/2$ challenge regarding the criterion for the absolute continuity of f on $[a, b]$. Let $\{(c_k, d_k)\}_{k=1}^n$ be a disjoint collection of open subintervals of (a, b) for which $\sum_{k=1}^n [d_k - c_k] < \delta$. For $1 \le k \le n$, let P_k be a partition of $[c_k, d_k]$. By the choice of δ in relation to the absolute continuity of f on $[a, b]$,

$$\sum_{k=1}^n TV(f_{[c_k, d_k]}, P_k) < \epsilon/2.$$

Take the supremum as, for $1 \le k \le n$, P_k vary among partitions of $[c_k, d_k]$, to obtain

$$\sum_{k=1}^n TV(f_{[c_k, d_k]}) \le \epsilon/2 < \epsilon.$$

We infer from (21) that, for $1 \le k \le n$, $TV(f_{[c_k, d_k]}) = TV(f_{[a, d_k]}) - TV(f_{[a, c_k]})$. Hence

$$\text{if } \sum_{k=1}^n [d_k - c_k] < \delta, \text{ then } \sum_{k=1}^n \left| TV(f_{[a, d_k]}) - TV(f_{[a, c_k]}) \right| < \epsilon. \tag{24}$$

Therefore the total variation function for f is absolutely continuous on $[a, b]$. □

Theorem 9 *Let the function f be continuous on the closed, bounded interval $[a, b]$. Then f is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{\text{Diff}_h f\}_{0<h\le 1}$ is uniformly integrable over $[a, b]$.*

Proof. First assume $\{\text{Diff}_h f\}_{0<h\le 1}$ is uniformly integrable over $[a, b]$. Let $\epsilon > 0$. Choose $\delta > 0$ for which

$$\int_E |\text{Diff}_h f| < \epsilon/2 \text{ if } m(E) < \delta \text{ and } 0 < h \le 1.$$

We claim that δ responds to the ϵ challenge regarding the criterion for f to be absolutely continuous. Indeed, let $\{(c_k, d_k)\}_{k=1}^n$ be a disjoint collection of open subintervals of (a, b) for which $\sum_{k=1}^n [d_k - c_k] < \delta$. For $0 < h \le 1$ and $1 \le k \le n$, by (14),

$$\text{Av}_h f(d_k) - \text{Av}_h f(c_k) = \int_{c_k}^{d_k} \text{Diff}_h f.$$

Therefore

$$\sum_{k=1}^n |\text{Av}_h f(d_k) - \text{Av}_h f(c_k)| \le \sum_{k=1}^n \int_{c_k}^{d_k} |\text{Diff}_h f| = \int_E |\text{Diff}_h f|,$$

where $E = \bigcup_{k=1}^n (c_k, d_k)$ has measure less than δ. Thus, by the choice of δ,

$$\sum_{k=1}^n |\text{Av}_h f(d_k) - \text{Av}_h f(c_k)| < \epsilon/2 \text{ for all } 0 < h \le 1.$$

Since f is continuous, take the limit as $h \to 0^+$ to obtain

$$\sum_{k=1}^n |f(d_k) - f(c_k)| \le \epsilon/2 < \epsilon.$$

Hence f is absolutely continuous.

To prove the converse, suppose f is absolutely continuous. The preceding theorem tells us that f is the difference of increasing absolutely continuous functions. We may therefore assume that f is increasing, so that the divided difference functions are nonnegative. To verify the uniformly integrability of $\{\text{Diff}_h f\}_{0<h\leq 1}$, let $\epsilon > 0$. We must show that there is a $\delta > 0$ such that for each measurable subset E of (a, b),

$$\int_E \text{Diff}_h f < \epsilon \text{ if } m(E) < \delta \text{ and } 0 < h \leq 1. \tag{25}$$

According to Theorem 11 of Chapter 2, a measurable set E is contained in a G_δ set G for which $m(G \sim E) = 0$. But every G_δ set is the intersection of a descending sequence of open sets. Moreover, every open set is the disjoint union of a countable collection of open intervals, and therefore every open set is the union of an ascending sequence of open sets, each of which is the union of a finite disjoint collection of open intervals. Therefore, by the continuity of integration, to verify (25) it suffices to find a $\delta > 0$ such that for $\{(c_k, d_k)\}_{k=1}^n$ a disjoint collection of open subintervals of (a, b),

$$\int_E \text{Diff}_h f < \epsilon/2 \text{ if } m(E) < \delta, \text{ where } E = \bigcup_{k=1}^n (c_k, d_k), \text{ and } 0 < h \leq 1. \tag{26}$$

Choose $\delta > 0$ as the response to the $\epsilon/2$ challenge regarding the criterion for the absolute continuity of f on $[a, b+1]$. By a change of variables for the Riemann integral and cancellation,

$$\int_u^v \text{Diff}_h f = \frac{1}{h} \cdot \int_0^h g(t)\,dt, \text{ where } g(t) = f(v+t) - f(u+t) \text{ for } 0 \leq t \leq 1 \text{ and } a \leq u < v \leq b.$$

Therefore, if $\{(c_k, d_k)\}_{k=1}^n$ is a disjoint collection of open subintervals of (a, b),

$$\int_E \text{Diff}_h f = \frac{1}{h} \cdot \int_0^h g(t)\,dt,$$

where

$$E = \bigcup_{k=1}^n (c_k, d_k) \text{ and } g(t) = \sum_{k=1}^n [f(d_k + t) - f(c_k + t)] \text{ for all } 0 \leq t \leq 1.$$

If $\sum_{k=1}^n [d_k - c_k] < \delta$, then, for $0 \leq t \leq 1$, $\sum_{k=1}^n [(d_k + t) - (c_k + t)] < \delta$, and therefore $g(t) < \epsilon/2$. Thus

$$\int_E \text{Diff}_h f = \frac{1}{h} \cdot \int_0^h g(t)\,dt < \epsilon/2.$$

Hence (26) is verified for this choice of δ. □

Remark *For a nondegenerate closed, bounded interval $[a, b]$, let $\mathcal{F}_{Lip}$, $\mathcal{F}_{AC}$, and $\mathcal{F}_{BV}$ denote the families of functions on $[a, b]$ that are Lipschitz, absolutely continuous, and of bounded variation, respectively. We have the following strict inclusions:*

$$\mathcal{F}_{Lip} \subsetneq \mathcal{F}_{AC} \subsetneq \mathcal{F}_{BV}. \tag{27}$$

44. Let f be Lipschitz on $\mathbf{R}$ and g be absolutely continuous on $[a, b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a, b]$.
45. Let f be absolutely continuous on $\mathbf{R}$ and g be absolutely continuous and strictly monotone on $[a, b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a, b]$.
46. Verify the assertions made in the final remark of this section.
47. Show that a function f is absolutely continuous on $[a, b]$ if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b),

$$\left|\sum_{k=1}^{n}[f(b_k) - f(a_k)]\right| < \epsilon \text{ if } \sum_{k=1}^{n}[b_k - a_k] < \delta.$$

6.5 INTEGRATING DERIVATIVES: DIFFERENTIATING INDEFINITE INTEGRALS

Let f be a continuous function on the closed, bounded interval $[a, b]$. In (14), take $a = u$ and $b = v$ to arrive at the following discrete formulation of the fundamental theorem of integral calculus:

$$\int_a^b \operatorname{Diff}_h f = \operatorname{Av}_h f(b) - \operatorname{Av}_h f(a).$$

Since f is continuous, the limit of the right-hand side as $h \to 0^+$ equals $f(b) - f(a)$. We now show that if f is absolutely continuous, then the limit of the left-hand side as $h \to 0^+$ equals $\int_a^b f'$ and thereby establish the fundamental theorem of integral calculus for the Lebesgue integral.[4]

Theorem 10 *Let the function f be absolutely continuous on the closed, bounded interval $[a, b]$. Then f is differentiable almost everywhere on (a, b), its derivative f' is integrable over $[a, b]$, and*

$$\int_b^a f' = f(b) - f(a). \tag{28}$$

Proof We infer from the discrete formulation of the fundamental theorem of integral calculus that

$$\lim_{n \to \infty} \left[\int_a^b \operatorname{Diff}_{1/n} f\right] = f(b) - f(a). \tag{29}$$

Theorem 8 tells us that f is the difference of increasing functions on $[a, b]$ and therefore, by Lebesgue's Theorem, is differentiable almost everywhere on (a, b). Therefore $\{\operatorname{Diff}_{1/n} f\}$ converges pointwise almost everywhere on (a, b) to f'. On the other hand, according to Theorem 9, $\{\operatorname{Diff}_{1/n} f\}$ is uniformly integrable over $[a, b]$. The Vitali Convergence Theorem (page 95) permits passage of the limit under the integral sign in order to conclude that

$$\lim_{n \to \infty} \left[\int_a^b \operatorname{Diff}_{1/n} f\right] = \int_a^b \lim_{n \to \infty} \operatorname{Diff}_{1/n} f = \int_a^b f'. \tag{30}$$

Formula (28) follows from (29) and (30). □

[4] This approach to the proof of the fundamental theorem of integral calculus for the Lebesgue integral is taken in a note by Patrick Fitzpatrick and Brian Hunt in which Theorem 9 is proven (see *www-users.math.umd.edu/~pmf/huntpmf*).

In the study of calculus, indefinite integrals are defined with respect to the Riemann integral. We here call a function f on a closed, bounded interval $[a, b]$ the **indefinite integral** of g over $[a, b]$ provided g is Lebesgue integrable over $[a, b]$ and

$$f(x) = f(a) + \int_a^x g \text{ for all } x \in [a, b]. \tag{31}$$

Theorem 11 *A function f on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.*

Proof First suppose f is absolutely continuous on $[a, b]$. For each $x \in (a, b]$, f is absolutely continuous over $[a, x]$ and hence, by the preceding theorem, in the case $[a, b]$ is replaced by $[a, x]$,

$$f(x) = f(a) + \int_a^x f'.$$

Thus f is the indefinite integral of f' over $[a, b]$.

Conversely, suppose that f is the indefinite integral over $[a, b]$ of g. For a disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b), if we define $E = \bigcup_{k=1}^n (a_k, b_k)$, then, by the monotonicity and additivity over domains properties of the integral,

$$\sum_{k=1}^n |f(b_k) - f(a_k)| = \sum_{k=1}^n \left| \int_{b_k}^{a_k} g \right| \le \sum_{k=1}^n \int_{b_k}^{a_k} |g| = \int_E |g|. \tag{32}$$

Let $\epsilon > 0$. Since $|g|$ is integrable over $[a, b]$, according to Proposition 23 of Chapter 4, there is a $\delta > 0$ such that $\int_E |g| < \epsilon$ if $E \subseteq [a, b]$ is measurable and $m(E) < \delta$. It follows from (32) that this same δ responds to the ϵ challenge regarding the criterion for f to be absolutely continuous on $[a, b]$. □

Corollary 12 *Let the function f be monotone on the closed, bounded interval $[a, b]$. Then f is absolutely continuous on $[a, b]$ if and only if*

$$\int_a^b f' = f(b) - f(a). \tag{33}$$

Proof Theorem 10 is the assertion that (33) holds if f is absolutely continuous, irrespective of any monotonicity assumption. Conversely, assume f is increasing and (33) holds. Let x belong to $[a, b]$. By the additivity over domains of integration,

$$0 = \int_a^b f' - [f(b) - f(a)] = \left\{ \int_a^x f' - [f(x) - f(a)] \right\} + \left\{ \int_x^b f' - [f(b) - f(x)] \right\}.$$

According to Corollary 4,

$$\int_a^x f' - [f(x) - f(a)] \le 0 \text{ and } \int_x^b f' - [f(b) - f(x)] \le 0.$$

If the sum of two nonnegative numbers is zero, then they both are zero. Therefore

$$f(x) = f(a) + \int_a^x f'.$$

Thus f is the indefinite integral of f'. The preceding theorem tells us that f is absolutely continuous. □

Lemma 13 *Let f be integrable over the closed, bounded interval $[a, b]$. Then*

$$f(x) = 0 \text{ for almost all } x \in [a, b] \tag{34}$$

if and only if

$$\int_{x_1}^{x_2} f = 0 \text{ for all } (x_1, x_2) \subseteq [a, b]. \tag{35}$$

Proof Clearly (34) implies (35). Conversely, suppose (35) holds. We claim that

$$\int_E f = 0 \text{ for all measurable sets } E \subseteq [a, b]. \tag{36}$$

Indeed, (36) holds for all open sets contained in (a, b) since integration is countably additive and every open set is the union of countable disjoint collection of open intervals. The continuity of integration then tells us that (36) also holds for all G_δ sets contained in (a, b) since every such set is the intersection of a countable descending collection of open sets. But every measurable subset of $[a, b]$ is of the form $G \sim E_0$, where G is a G_δ subset of (a, b) and $m(E_0) = 0$ (see page 40). We conclude from the additivity over domains of integration that (36) is verified. Define

$$E^+ = \{x \in [a, b] \mid f(x) \geq 0\} \text{ and } E^- = \{x \in [a, b] \mid f(x) \leq 0\}.$$

These are two measurable subsets of $[a, b]$ and therefore, by (36),

$$\int_a^b f^+ = \int_{E^+} f = 0 \text{ and } \int_a^b (-f^-) = -\int_{E^-} f = 0.$$

According to Proposition 9 of Chapter 4, a nonnegative integrable function with zero integral must vanish almost everywhere on its domain. Thus f^+ and f^- vanish almost everywhere on $[a, b]$ and hence so does f. □

Theorem 14 *Let f be integrable over the closed, bounded interval $[a, b]$. Then*

$$\frac{d}{dx}\left[\int_a^x f\right] = f(x) \text{ for almost all } x \in (a, b). \tag{37}$$

Proof Define the function F on $[a, b]$ by $F(x) = \int_a^x f$ for all $x \in [a, b]$. Theorem 18 tells us that since F is an indefinite integral, it is absolutely continuous. Therefore, by Theorem 10, F

is differentiable almost everywhere on (a, b) and its derivative F' is integrable. According to the preceding lemma, to show that the integrable function $F' - f$ vanishes almost everywhere on $[a, b]$ it suffices to show that its integral over every closed subinterval of $[a, b]$ is zero. Let $[x_1, x_2]$ be contained in $[a, b]$. According to Theorem 10, in the case $[a, b]$ is replaced by $[x_1, x_2]$, and the linearity and additivity over domains properties of integration,

$$\int_{x_1}^{x_2} [F' - f] = \int_{x_1}^{x_2} F' - \int_{x_1}^{x_2} f = F(x_2) - F(x_1) - \int_{x_1}^{x_2} f$$

$$= \int_a^{x_2} f - \int_a^{x_1} f - \int_{x_1}^{x_2} f = 0.$$

□

A function of bounded variation is said to be **singular** provided its derivative vanishes almost everywhere. The Cantor-Lebesgue function is a non-constant singular function. We infer from Theorem 10 that an absolutely continuous function is singular if and only if it is constant. Let f be of bounded variation on $[a, b]$. According to Corollary 6, f' is integrable over $[a, b]$. Define

$$g(x) = \int_a^x f' \text{ and } h(x) = f(x) - \int_a^x f' \text{ for all } x \in [a, b],$$

so that

$$f = g + h \text{ on } [a, b].$$

According to Theorem 11, the function g is absolutely continuous. We infer from Theorem 14 that the function h is singular. The above decomposition of a function of bounded variation f as the sum $g + h$ of two functions of bounded variation, where g is absolutely continuous and h is singular, is called a **Lebesgue decomposition** of f.

PROBLEMS

48. The Cantor-Lebesgue function φ is continuous and increasing on $[0, 1]$. Conclude from Theorem 10 that φ is not absolutely continuous on $[0, 1]$. Compare this reasoning with that proposed in Problem 40.

49. Let f be continuous on $[a, b]$ and differentiable almost everywhere on (a, b). Show that

$$\int_a^b f' = f(b) - f(a)$$

if and only if

$$\int_a^b \left[\lim_{n\to\infty} \text{Diff}_{1/n} f\right] = \lim_{n\to\infty} \left[\int_a^b \text{Diff}_{1/n} f\right].$$

50. Let f be continuous on $[a, b]$ and differentiable almost everywhere on (a, b). Show that if $\{\text{Diff}_{1/n} f\}$ is uniformly integrable over $[a, b]$, then

$$\int_a^b f' = f(b) - f(a).$$

6.6 CONVEX FUNCTIONS

Throughout this section (a, b) is an open interval that may be bounded or unbounded.

Definition *A real-valued function φ on (a, b) is said to be* **convex** *provided for each pair of points x_1, x_2 in (a, b) and each λ with $0 \leq \lambda \leq 1$,*

$$\varphi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2). \tag{38}$$

If we look at the graph of φ, the convexity inequality can be formulated geometrically by saying that each point on the chord between $(x_1, \varphi(x_1))$ and $(x_2, \varphi(x_2))$ is above the graph of φ.

Observe that for two points $x_1 < x_2$ in (a, b), each point x in (x_1, x_2) may be expressed as

$$x = \lambda x_1 + (1-\lambda)x_2 \text{ where } \lambda = \frac{x_2 - x}{x_2 - x_1}.$$

Thus the convexity inequality may be written as

$$\varphi(x) \leq \left[\frac{x_2 - x}{x_2 - x_1}\right]\varphi(x_1) + \left[\frac{x - x_1}{x_2 - x_1}\right]\varphi(x_2) \text{ for } x_1 < x < x_2 \text{ in } (a, b).$$

Regathering terms, this inequality may also be rewritten as

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x} \text{ for } x_1 < x < x_2 \text{ in } (a, b). \tag{39}$$

Therefore convexity may also be formulated geometrically by saying that for $x_1 < x < x_2$, the slope of the chord from $(x_1, \varphi(x_1))$ to $(x, \varphi(x))$ is no greater than the slope of the chord from $(x, \varphi(x))$ to $(x_2, \varphi(x_2))$.

Proposition 15 *If φ is differentiable on (a, b) and its derivative φ' is increasing, then φ is convex. In particular, φ is convex if it has a nonnegative second derivative φ'' on (a, b).*

Proof Let x_1, x_2 be in (a, b) with $x_1 < x_2$, and let x belong to (x_1, x_2). We must show that

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.$$

However, apply the Mean Value Theorem to the restriction of φ to each of the intervals $[x_1, x]$ and $[x, x_2]$ to choose points $c_1 \in (x_1, x)$ and $c_2 \in (x, x_2)$ for which

$$\varphi'(c_1) = \frac{\varphi(x) - \varphi(x_1)}{x - x_1} \text{ and } \varphi'(c_2) = \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.$$

Thus, since φ' is increasing,

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} = \varphi'(c_1) \leq \varphi'(c_2) = \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.$$

□