# Differentiation and Integration

Contents



The fundamental theorems of integral and differential calculus, with respect to the Riemann integral, are the workhorses of calculus. In this chapter we formulate these two theorems for the Lebesgue integral. For a function $f$ on the closed, bounded interval $[a, b]$, when is

$$\int_a^b f' = f(b) - f(a)? \tag{i}$$

Assume $f$ is continuous. Extend $f$ to take the value $f(b)$ on $(b, b+1]$, and for $0 < h \leq 1$, define the divided difference function $\mathrm{Diff}_h f$ and average value function $\mathrm{Av}_h f$ on $[a, b]$ by

$$\mathrm{Diff}_h f(x) = \frac{f(x+h) - f(x)}{h} \text{ and } \mathrm{Av}_h f(x) = \frac{1}{h}\int_x^{x+h} f(t)\,dt \text{ for all } x \text{ in } [a, b].$$

A change of variables and cancellation provides the discrete formulation of (i) for the Riemann integral:

$$\int_a^b \mathrm{Diff}_h f = \mathrm{Av}_h f(b) - \mathrm{Av}_h f(a).$$

The limit of the right-hand side as $h \to 0^+$ equals $f(b) - f(a)$. We prove a striking theorem of Henri Lebesgue which tells us that a monotone function on $(a, b)$ has a finite derivative almost everywhere. We then define what it means for a function to be absolutely continuous and prove that if $f$ is absolutely continuous, then $f$ is the difference of monotone functions and the collection of divided differences, $\{\mathrm{Diff}_h f\}_{0<h\leq 1}$, is uniformly integrable. Therefore, by the Vitali Convergence Theorem, (i) follows for $f$ absolutely continuous by taking the limit as $h \to 0^+$ in its discrete formulation. If $f$ is monotone and (i) holds, we prove that $f$ must be absolutely continuous. From the integral form of the fundamental theorem, (i), we obtain the differential form, namely, if $f$ is Lebesgue integrable over $[a, b]$, then

$$\frac{d}{dx}\left[\int_a^x f\right] = f(x) \text{ for almost all } x \text{ in } [a, b]. \tag{ii}$$

## 6.1 CONTINUITY OF MONOTONE FUNCTIONS

Recall that a function is defined to be monotone if it is either increasing or decreasing. Monotone functions play a decisive role in resolving the question posed in the preamble. There are two reasons for this. First, a theorem of Lebesgue (page 112) asserts that a monotone function on an open interval is differentiable almost everywhere. Second, a theorem of Jordan (page 117) tells us that a very general family of functions on a closed, bounded interval, those of bounded variation, which includes Lipschitz functions, may be expressed as the difference of monotone functions and therefore they also are differentiable almost everywhere on the interior of their domain. In this brief preliminary section we consider continuity properties of monotone functions.

**Theorem 1** *Let $f$ be a monotone function on the open interval $(a, b)$. Then $f$ is continuous except possibly at a countable number of points in $(a, b)$.*

**Proof** Assume $f$ is increasing. Furthermore, assume $(a, b)$ is bounded and $f$ is increasing on the closed interval $[a, b]$. Otherwise, express $(a, b)$ as the union of an ascending sequence of open, bounded intervals, the closures of which are contained in $(a, b)$, and take the union of the discontinuities in each of this countable collection of intervals. For each $x_0 \in (a, b)$, $f$ has a limit from the left and from the right at $x_0$. Define

$$f(x_0^-) = \lim_{x \to x_0^-} f(x) = \sup \{f(x) \mid a < x < x_0\},$$

$$f(x_0^+) = \lim_{x \to x_0^+} f(x) = \inf \{f(x) \mid x_0 < x < b\}.$$

Since $f$ is increasing, $f(x_0^-) \le f(x_0^+)$. The function $f$ fails to be continuous at $x_0$ if and only if $f(x_0^-) < f(x_0^+)$, in which case we define the open "jump" interval $J(x_0)$ by

$$J(x_0) = \{y \mid f(x_0^-) < y < f(x_0^+)\}.$$

Each jump interval is contained in the bounded interval $[f(a), f(b)]$ and the collection of jump intervals is disjoint. Therefore, for each natural number $n$, there are only a finite number of jump intervals of length greater than $1/n$. Thus the set of points of discontinuity of $f$ is the union of a countable collection of finite sets and therefore is countable. □

**Proposition 2** *Let $C$ be a countable subset of the open interval $(a, b)$. Then there is an increasing function on $(a, b)$ that is continuous only at points in $(a, b) \sim C$.*

**Proof** If $C$ is finite the proof is clear. Assume $C$ is countably infinite. Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of $C$. Define the function $f$ on $(a, b)$ by setting[1]

$$f(x) = \sum_{\{n \mid q_n \le x\}} \frac{1}{2^n} \text{ for all } a < x < b.$$

[1] We use the convention that a sum over the empty-set is zero.

Since a geometric series with a ratio less than 1 converges, $f$ is properly defined. Moreover,

$$\text{if } a < u < v < b, \text{ then } f(v) - f(u) = \sum_{\{n \mid u < q_n \le v\}} \frac{1}{2^n}. \tag{1}$$

Thus $f$ is increasing. Let $x_0 = q_k$ belong to $C$. Then, by (1),

$$f(x_0) - f(x) \ge \frac{1}{2^k} \text{ for all } x < x_0.$$

Therefore $f$ fails to be continuous at $x_0$. Now let $x_0$ belong to $(a, b) \sim C$. Let $n$ be a natural number. There is an open interval $I$ containing $x_0$ for which $q_n$ does not belong to $I$ for $1 \le k \le n$. We infer from (1) that $|f(x) - f(x_0)| < 1/2^n$ for all $x \in I$. Therefore $f$ is continuous at $x_0$. □

### PROBLEMS

1. Let $C$ be a countable subset of the nondegenerate closed, bounded interval $[a, b]$. Show that there is an increasing function on $[a, b]$ that is continuous only at points in $[a, b] \sim C$.
2. Show that there is a strictly increasing function on $[0, 1]$ that is continuous only at the irrational numbers in $[0, 1]$.
3. Let $f$ be a monotone function on a subset $E$ of $\mathbf{R}$. Show that $f$ is continuous except possibly at a countable number of points in $E$.
4. Let $E$ be a subset of $\mathbf{R}$ and $C$ a countable subset of $E$. Is there a monotone function on $E$ that is continuous only at points in $E \sim C$?

## 6.2 DIFFERENTIABILITY OF MONOTONE FUNCTIONS: LEBESGUE'S THEOREM

A closed, bounded interval $[c, d]$ is said to be nondegenerate provided $c < d$.

**Definition** *A collection $\mathcal{F}$ of closed, bounded, nondegenerate intervals is said to cover a set $E$* **in the sense of Vitali** *provided for each point $x$ in $E$ and $\epsilon > 0$, there is an interval $I$ in $\mathcal{F}$ that contains $x$ and has $\ell(I) < \epsilon$.*

**The Vitali Covering Lemma** *Let $E$ be a set of finite outer measure and $\mathcal{F}$ a collection of closed, bounded intervals that covers $E$ in the sense of Vitali. Then for each $\epsilon > 0$, there is a finite disjoint subcollection $\{I_k\}_{k=1}^n$ of $\mathcal{F}$ for which*

$$m^*\left[E \sim \bigcup_{k=1}^{n} I_k\right] < \epsilon. \tag{2}$$

**Proof** Since $m^*(E) < \infty$, there is an open set $\mathcal{O}$ containing $E$ for which $m(\mathcal{O}) < \infty$. Because $\mathcal{F}$ is a Vitali covering of $E$, we may assume that each interval in $\mathcal{F}$ is contained in $\mathcal{O}$. By the countable additivity and monotonicity of measure,

$$\text{if } \{I_k\}_{k=1}^{\infty} \subseteq \mathcal{F} \text{ is disjoint, then } \sum_{k=1}^{\infty} \ell(I_k) \le m(\mathcal{O}) < \infty. \tag{3}$$