

Completeness in the set of Real numbers.

$\{Q, \mathbb{R}\} \rightarrow$ ordered fields,
whereas $N, I \rightarrow$ fails.

Now, we shall go through another property, called completeness / order-completeness, which is possessed by \mathbb{R} but not by Q .

→ This property not only distinguishes \mathbb{R} from Q , but together with the ordered field property.

i.e. \mathbb{R} ; i.e., the set of real numbers is the only set which is a complete order field.

Order Completeness in \mathbb{R} .

O-C Every non-empty set of real numbers which is bounded above has the supremum (lub) in \mathbb{R} .
 → In other words, the set of upperbounds of a nonempty set of real numbers bounded above has the smallest number.

→ If S is a set of real numbers which is bounded above, then by considering the set

$$T = \{x : -x \in S\}$$

we state the completeness property in an alternate way as given below:

O-C_(alt) Every nonempty set of real numbers which is bounded below has the infimum (or glb) in \mathbb{R} .

→ In other words, the set of lower bounds of a nonempty set of real numbers bounded below has the greatest number.

Theorem 1. The set of rational numbers is not order complete.

Proof.: In order to show that the set of rational numbers does not possess the property of completeness, it is sufficient to show that there exists a nonempty set S of rational numbers (a subset of \mathbb{Q}) which is bounded above but does not have a supremum (lub) in \mathbb{Q} ; that is, no rational number exists which can be a supremum of S .

Let S be a subset (of \mathbb{Q}) / better to say:

Let S be a set (subset of \mathbb{Q}) of all those positive rational numbers whose square is less than 2, i.e.,

$$S = \{x; \exists x \in \mathbb{Q}, x^2 < 2\}.$$

Clearly, $1 \in S$ and $1^2 = 1 < 2$. Hence $1 \in S$. So S is nonempty. (This shows that we do not deal with any trivial or vague set).

Clearly 2 is an upper bound of S . Hence S is bounded above.

Thus, S is a nonempty set of rational numbers, bounded above.

Let if possible, K be its lub, K is rational. Clearly K is positive. Also, by the law of Trichotomy property, one of the following holds good: $K^2 < 2$, $K^2 = 2$, $K^2 > 2$.

Let $K^2 < 2$. Then consider the positive rational number

$$y = \frac{y+3K}{3+2K}.$$

Then

$$K-y = \frac{2(K^2-2)}{3+2K} < 0 \Rightarrow y > K.$$

Also, $2-y^2 = \frac{2-K^2}{(3+2K)^2} > 0 \Rightarrow \underline{y^2 < 2} \Rightarrow \text{yes.}$

Thus, the member y of S is greater than K , which shows that K can not be upper bound, which is a contradiction to our assumption that K is the lub of S .

$K^2 = 2$: K can not be rational. So this case do not arise.

Finally, suppose that $K^2 > 2$. Considering the positive rational number y , defined as above, we can show that $y < K$ and $y^2 > 2$.

Hence, there exists an upper bound y of S , smaller than K , which is a contradiction.

Thus, none of the three possible cases holds, indicating that our supposition is wrong. Thus, no rational number exists which can be a lub of S .

Archimedean Property of Real Numbers

Theorem: The real number field is Archimedean, i.e., if a and b are any two positive real numbers, then there exists a positive integer n such that $na > b$.

Proof: Let a and b be any two positive real numbers. Let us suppose, if possible, that for all the integers n , i.e., $n \in \mathbb{N}$, we have $na \leq b$.

Thus, the set $S = \{na; n \in \mathbb{N}\}$ is bounded above. Note that b is the upper bound. By the completeness property of the order field of real numbers, the set S may have a supremum, say M .

Then

$$na \leq M, \quad \forall n \in \mathbb{N}.$$

$$\Rightarrow (n+1)a \leq M \quad \forall n \in \mathbb{N}.$$

$$\Rightarrow na \leq M-a \quad \forall n \in \mathbb{N},$$

that is, $M-a$ is an upper bound of S .

Thus, a member $M-a$ is less than the supremum M (lub) is an upper bound of S , which is a contradiction. Hence $na > b$. The proof is complete.

Corollary: If a is a positive real number and b is any real number, then there exists a positive integer n such that $na > b$.

Corollary: For any real number b , \exists a positive integer n such that $n > b$.

Corollary: For any $\epsilon > 0$, \exists a positive integer n s.t $\frac{1}{n} < \epsilon$.

choose $b = \frac{1}{\epsilon}$. work done.