A Brief Overview on Population Dynamics

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Fundamentals of Mathematical Ecology/Biology

- Provides an introduction to Classical and Modern Mathematical Models, Methods and issues in population dynamics.
- Devoted to simple models for the sake of tractability.
- Topics covered include single species models, interacting populations that include predation.
- Suitable for beginning researchers in Mathematical Modelling.

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Some useful results from ODE

- A general first order initial value problem is given by

$$y' = f(t, y), y(0) = y_0.$$

- —a non-autonomous differential equation due to explicit involvement of the independent variable, t, in the right hand side.
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Unless and otherwise stated, we assume that all the differential equations satisfy the Picard's theorem. Hence every initial value problem admits a unique solution.

Theorem (Picard's Theorem)

If $g(t,u)$ is a continuous function of t and u in a closed and bounded region R containing a point (t_0,u_0) and satisfies the Lipschitz condition in R then there exists a unique solution $u(t)$ to the initial value problem $u' = g(t,u), u(t_0) = u_0$ defined on an interval J containing t_0.

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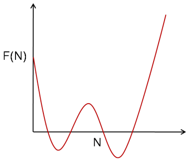

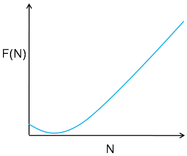

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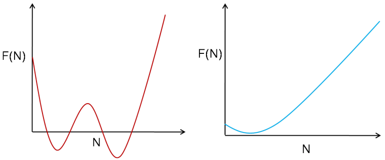

Remark

These solutions are also some times called as equilibrium points, critical points, stationery points, rest points or fixed points.

- If N^* is an equilibrium solution of the differential equation $N' = f(N)$ then $N(t) = N^*$ is the unique (constant) solution of the initial value problem (IVP) $N' = f(N), N(t_0) = N^*$.
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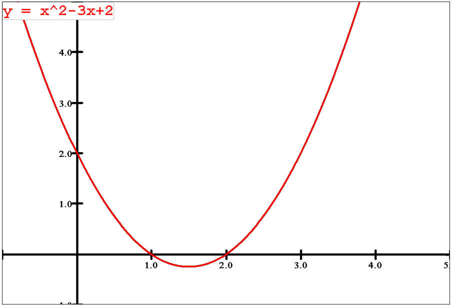

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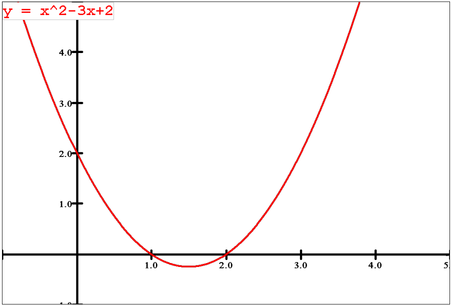

- It is easy to verify that $x^* = 2$ and $x^* = 1$ satisfy the differential equation $x' = x^2 - 3x + 2$.

Definition (Stability)

An equilibrium solution N^* is said to be Lyapunov stable, if for any given $\epsilon > 0$ there exists a $\delta > 0$ (depending on ϵ) such that, for all initial conditions $N(t_0) = N_0$ satisfying $|N_0 - N^*| < \delta$, we have $|N(t) - N^*| < \epsilon$ for all $t > t_0$.

Alternatively, we say that an equilibrium solution is said to be stable if solutions starting close to equilibrium solution (in a δ neighborhood) remain in its ϵ neighborhood for all future times.

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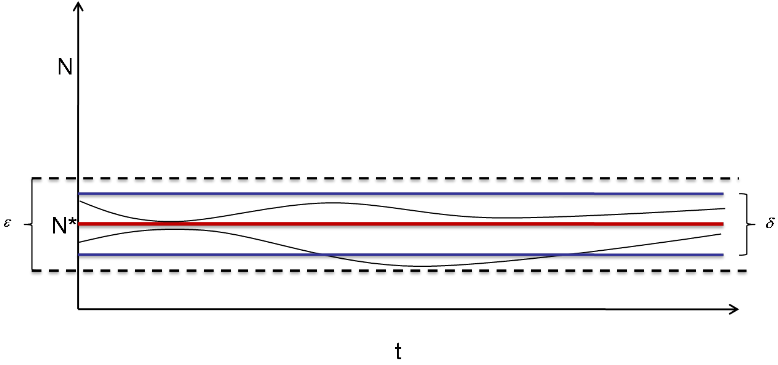
Stability - Pictorial Representation



Definition (Asymptotical Stability)

An equilibrium solution N^* is said to be asymptotically stable if

- it is stable
- if there exists a $\rho > 0$ such that for all N_0 such that $|N_0 - N^*| < \rho \Rightarrow \lim_{t\to\infty} |N(t) - N^*| = 0$.

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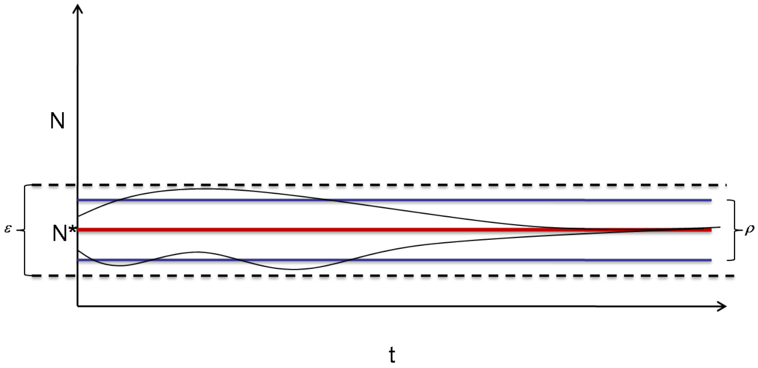

Definition (Unstable Solution)

A solution of the system

$$\frac{dN}{dt} = f(N).$$

is said to be unstable if it is not stable.

Theorem (A Useful Theorem in One-dimensional Space)

Suppose that N^* is an equilibrium point of the differential equation $N' = f(N)$, where $f(N)$ is assumed to be a continuously differentiable function with $f'(N^*) \neq 0$. Then the equilibrium point N^* is asymptotically stable if $f'(N^*) < 0$, and unstable if $f'(N^*) > 0$.

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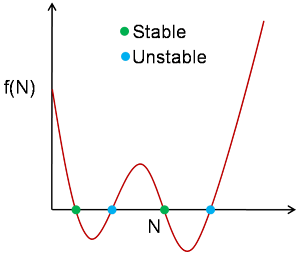

Single Species Dynamics

Populations...

- Population will be a primitive concept for us.
  - It concerns groups of living organisms (plants, animals, micro-organisms..) which are composed of individuals with a similar dynamical behavior.
  - We postulate that every living organism has arisen from another one and populations reproduce.
  - Note: we will study populations and not the individuals.
- Populations change in size, they grow or decrease due to birth, death, migration.

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The Basic Framework

- We want to study laws that govern population changes in space and time.
- We begin by restricting our study to how populations change in time. We call these changes dynamical. Our basic framework is
- First: a population is described by its number of individuals (in some cases, however, by the biomass).
  - We will here study unstructured populations and leave aside structured populations arises and are important. Here structure could be classes of age, size, sex...
- Second: we need to describe the time variation of the population. We will use (ordinary) derivatives for this purpose. Alternatively, we could also work with stochastic processes or discrete-time formulations...
- Three: we need to know what causes these time variations. Which biological processes. Then we have to translate this into (convenient) mathematical language how these biological processes affect the time-changes of the population.

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Dynamics of a single species

- $N(t)$ represents total number or density of a population in an environment.
- $\frac{dN}{dt}$ stands for rate of change in the entire population.
- $\frac{1}{N}\frac{dN}{dt}$ represents per capita rate of change in the entire population. (Change in the total population due to an individual.)
- To start with, we assume that the population changes due to *births and deaths* only.
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- Thus the governing dynamic equation for the population is

$$\frac{dN}{dt} = rN$$

where $r = b - d$ called as *intrinsic growth rate*. This model is called *exponential model* or *Malthusian model*.

- If the initial population at time t_0, $N(t_0) = N_0$, the solution of this differential equation is given by

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(1766-1834)

Malthusian's Model, $\frac{dN}{dt} = rN, N(t_0) = N_0$

- Per capita growth rate, $\frac{1}{N}\frac{dN}{dt}$, is always constant $(b - d)$.
- The growth rate of the population is always increasing (decreasing) if $r > 0 (r < 0)$.
- The population grows (decays) exponentially from the initial value N_0 since $N(t) = N_0 e^{rt}$. The population will remain constant only when births and deaths balance each other, i.e., $b = d$ or $r = 0$.

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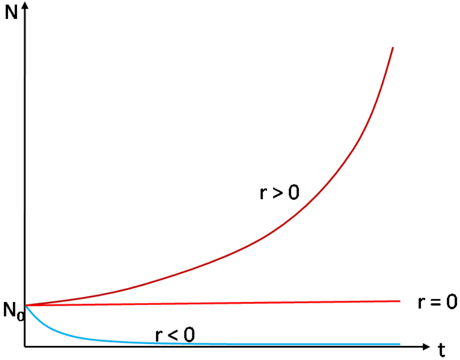

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- From the model we observe that the population either blows up to infinity or decays to zero exponentially which is not realistic.
- This calls for a modification in the model. The present model assumes that the per capita growth rate is independent of the population.
- It is more realistic to assume that the per capita growth rate to be a function of total population in view of the fact that the population always has to share the limited food resources which naturally limits their growth.
- Before going into this, some examples:

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Exponential Growth

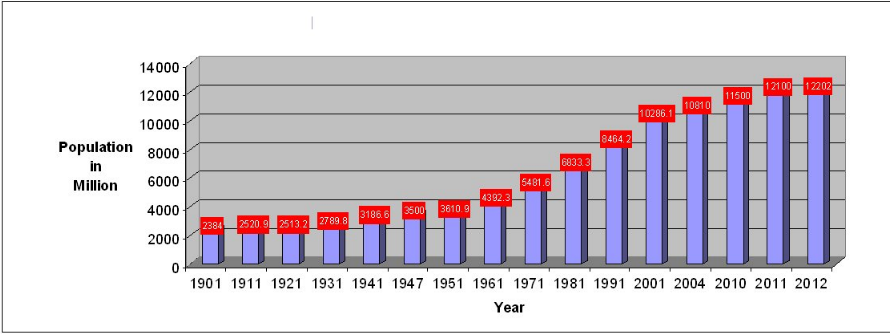

Figure: The population of India. Until 1951-1961, the growth is well approximated by an exponential curve.

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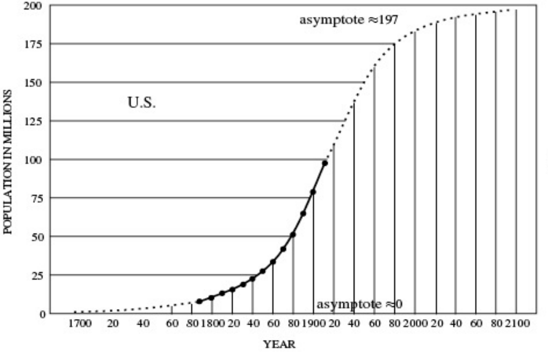

Figure: The population of USA. Until 1920, the growth is well approximated by an exponential curve.

Modified Model

- Hence we assume that the per capita growth rate of the population is linearly decreasing function of the total population,given by

$$\frac{1}{N}\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)$$

where K is called carrying capacity which represents the total population the environment can support.

- Observe that the per capita growth rate continuously reduces from r as the population N increases from zero and it becomes zero when the population reaches K. This seems reasonable as resources are always limited and the population are controlled by these resources.

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Logistic Model

The modified model representing growth in a species is given by

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), N(t_0) = N_0$$

which is called as Logistic Model or Verhulst Model.

Logistic Model

Figure: P-F. Verhulst, first introduced the logistic equation in 1838. On the right side, Raymond Pearl, who “rediscovered” Verhulst’s work.

Parabolic Population Growth: $f(N) = rN\left(1 - \frac{N}{K}\right)$

- Let us analyze the Logistic model in the light of the theorem done previously.
- We have

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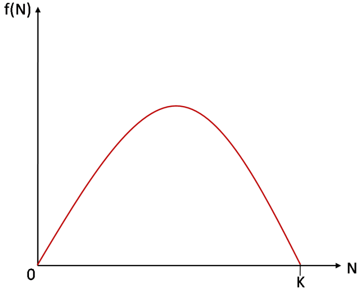

Analysis of Logistic Equation $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),\ N(t_0) = N_0$

$$f(N) = 0 \Rightarrow N = 0 \text{ or } K.$$

Logistic equation admits two equilibrium points given by $N_1 = 0, N_2 = K$

$$f'(N) = r\left(1 - \frac{2N}{K}\right), \quad f'(N_1) = r > 0, \quad f'(N_2) = -r < 0$$

The trivial equilibrium, $N_1 = 0$ is unstable and $N_2 = K$ is asymptotically stable.

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- $\left(\frac{1}{N} + \frac{1}{K-N}\right) dN = rdt.$
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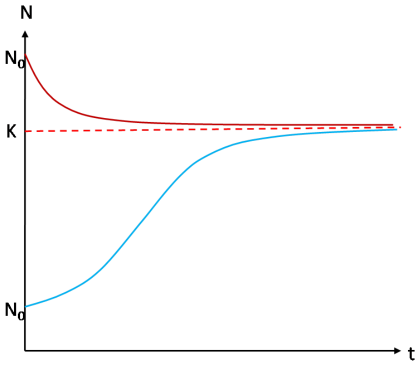
Logistic Growth



Logistic Growth

Simulation

Glory

- Simple and its solvable.
- Allows us to introduce the concept of carrying capacity.
- A good approximation in several cases.

Misery

- Too simple
- Does not model more complex biological facts

Why one should like the logistic equation?

- Minimal model using which we can build more complex/sophisticated ones.

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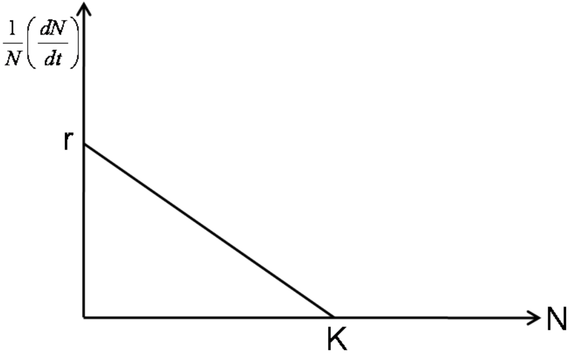
Further modification to the Logistic equation

- Logistic equation assumes that Per capita growth large when the population is small.



Further modification to the Logistic equation ...

- If the population is small there may not be any interaction at all among the population.
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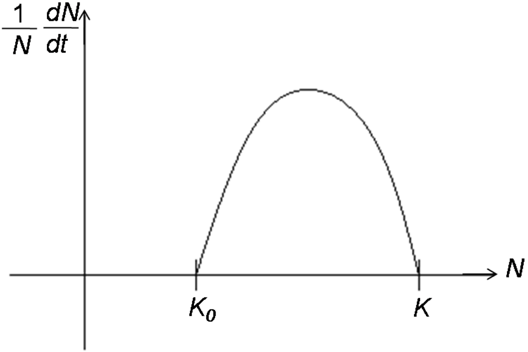
Per-capita Growth With Allee Effect



Equation with Allee Effect

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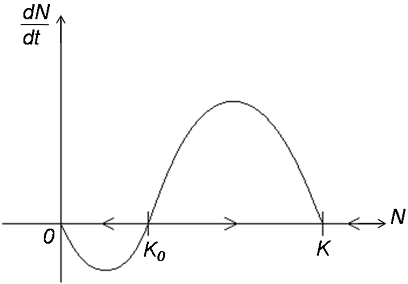

Simulation

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Interacting Populations

Interacting populations

- Till now we have discussed about the dynamics of a single species with and without harvesting.
- Applied bifurcation analysis to derive conclusions on maximum sustainable yield.
- Present discussion is devoted to study the dynamics of two interacting population.
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Predator-Prey Model development

- We assume coexistence of predators and prey in an environment.
- $N(t)$ - number or density of prey.
- $P(t)$ - number or density of predator.
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- There are encounters between prey and predator.
- They result in consumption of prey by the predator.
- Number of encounters is proportional their densities.
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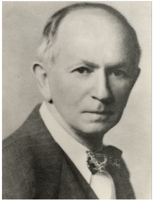
Lotka-Volterra

Vito Volterra
(1860 - 1940)



Alfred James Lotka
(1880 - 1949)

Vito Volterra (1860-1940), an Italian mathematician, proposed the equation now known as the Lotka-Volterra one to understand a problem proposed by his future son-in-law, Umberto d'Ancona, who tried to explain oscillations in the quantity of predator fishes captured at the certain ports of the Adriatic sea.

Alfred Lotka (1880-1949), was an USA mathematician and chemist, born in Ukraine, who tried to transpose the principles of physical-chemistry to biology. He published his results in a book called "Elements of Physical Biology". His results are independent from the work of Volterra.

Lotka - Volterra equations: $N' = rN - cNP, P' = bNP - mp.$

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- Find if there are any equilibrium points for the system and investigate their nature.
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A few definitions

- Consider an autonomous system
  $x' = F(x,y), \;\; y' = G(x,y)$
- Equilibrium points of this system are the points satisfying
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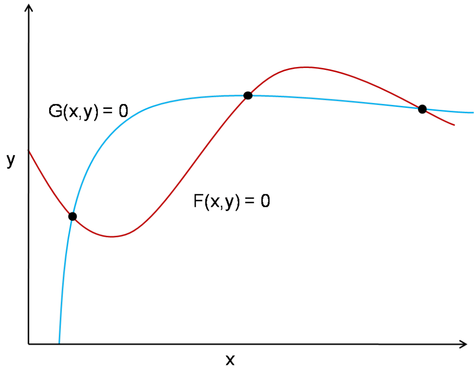
Isoclines



Equilibrium Points of Lotka-Volterra Equations

Isoclines: $(r - cP)N = 0, \quad (bN - m)P = 0$

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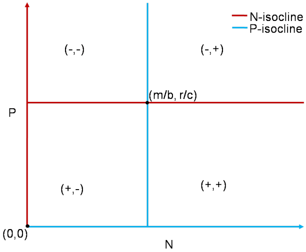
Isocline Figure



$N' = rN - cNP, \quad P' = bNP - mP$: Analysis near $(0,0)$

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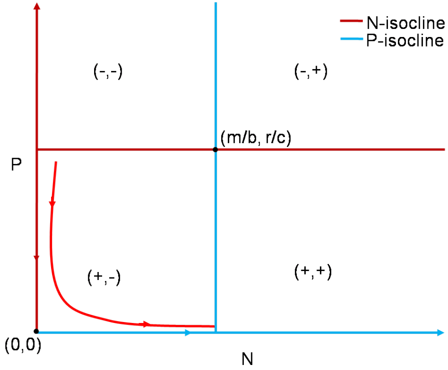

$N' = rN - cNP, \quad P' = bNP - mP$: Analysis near $(\frac{m}{b}, \frac{r}{c})$

- Define $u = N - N_2, \quad v = P - P_2$.
- This transforms the original system to $u' = -\frac{mc}{b}v - cuv, \quad v' = \frac{rb}{c}u + buv$.
- Let (N_0, P_0) be in the vicinity of (N_2, P_2).
- Since u, v are small we can neglect their product terms and hence we obtain $\frac{dv}{du} \approx -\frac{rb^2}{mc^2}\frac{u}{v}$
- $mc^2 vdv + rb^2 udu = 0$
- $rb^2u^2 + mc^2v^2 = c^2 \Rightarrow rb^2(N - N_2)^2 + mc^2(P - P_2)^2 = C^2$, represent ellipses around (N_2, P_2).

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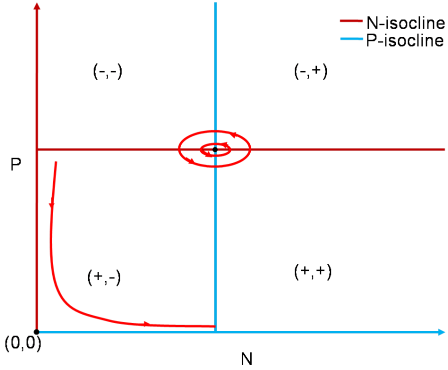

Global behaviour of $N' = rN - cNP, \quad P' = bNP - mP$

- Eliminating t and rearranging the system we obtain $\frac{(r-cP)dP}{P} = \frac{(bN-m)dN}{N}$.
- Upon integration, we obtain $P^r e^{-cP} = KN^{-m}e^{bN}$.
- represents ovals about (N_2, P_2) in anti clockwise direction.

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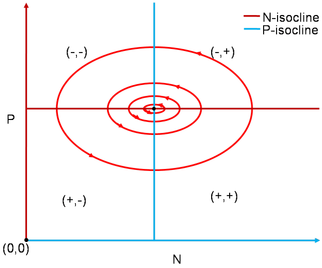

Thus the considered system admits periodic solutions.

Periodic Solutions

- Let T be the period of a solutions.
- Consider the equation $\frac{dN}{dt} = rN - cNP$
- Separating the variables and integrating over a time period T we obtain $\int\limits_{t_0}^{t_0+T} \frac{dN}{N} = \int\limits_{t_0}^{t_0+T} (r - cP)dt$

$$\Rightarrow ln\left[\frac{N(t_0+T)}{N(t_0)}\right] = rT - c\int\limits_{t_0}^{t_0+T} Pdt$$

- Thus $rT - c\int\limits_{t_0}^{t_0+T} Pdt = 0$
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Misery

- As Lotka-Volterra model exhibits orbital stability, once you are on a certain orbit in the phase space, it has certain amplitude and period.
- If we perturb this orbit, the system will stay on a new orbit, with different amplitude and period.
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Some References

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THANK YOU