

Normed Linear Spaces

Definition

Let X be a linear space over K . A norm on X is a function $\|\cdot\|: X \rightarrow K$ such that for $x, y \in X$ and $k \in K$,

- (i) $\|x\| \geq 0$, and $\|x\| = 0$ iff $x = 0$
- (ii) $\|x+y\| \leq \|x\| + \|y\|$; and
- (iii) $\|kx\| = |k| \cdot \|x\|$.

A normed linear space (nls) X is a linear space with a norm $\|\cdot\|$ on it.

- If we define for $x, y \in X$, $d(x, y) = \|x-y\|$, then d is easily seen to be a metric on X . (Verify).
- The function $\|\cdot\|: X \rightarrow K$ is uniformly continuous on X , since $|\|x\| - \|y\|| \leq \|x-y\|$ for all $x, y \in X$.
- The operations ~~of~~ of addition and scalar multiplication in X are jointly continuous in the sense that whenever $x_n \rightarrow x$ and $y_n \rightarrow y$ in X and $k_n \rightarrow k$ in K , then $x_n + y_n \rightarrow x+y$ and $k_n x_n \rightarrow kx$ in X .
- An important relation between the metric and the linear structures of a nls is obtained by noting that for every $x \in X$ and $r > 0$, the open ball

$$U(x, r) = \{y \in X ; \|x-y\| < r\} \quad (\text{looks like } d(r, y) < r)$$

and its closure

$$\bar{U}(x, r) = \{y \in X ; \|x-y\| \leq r\} \quad (\text{looks like } d(r, y) \leq r).$$

are both convex subsets of X .

Theorem 1. Let X be a nls.

- (a) If $E_1 \subset X$ is open and $E_2 \subset X$, then $E_1 + E_2$ is open;
- (b) If $E \subset X$ is convex, then so are E° and \bar{E} .
- (c) If $Y \subset X$ is a subspace of X , then $Y \neq X$ iff $Y^\circ = \emptyset$.
- (d) If $f: X \rightarrow K$ is a linear functional, then either $f=0$ or f maps open sets in X onto open sets in K .

Proof (a). Let $x_2 \in E_2$. If $x_1 \in E_1$, then since E_1 is open, there is $r > 0$ such that $U(x_1, r) \subset E_1$. Now,

$$U(x_1 + x_2, r) = U(x_1, r) + x_2 \subset E_1 + x_2$$

so that $E_1 + x_2$ is open. Since a union of open sets is open,
 ↳ translation property

$$\text{then } E_1 + E_2 = \bigcup \{ E_1 + x_2 ; x_2 \in E_2 \}$$

is open.

(b). Since E° is open, it follows that for $0 < r < 1$, $rE^\circ + (1-r)E^\circ$ is open; also it is contained in E because E is convex (assumed). This shows that

$$rE^\circ + (1-r)E^\circ \subset E^\circ.$$

Thus E° is convex.

Next, let $x, y \in \bar{E}$. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in E such that $x_n \rightarrow x$ and $y_n \rightarrow y$. If $0 < r < 1$, then since E is convex, $r x_n + (1-r)y_n \in E$ and tends $rx + (1-r)y$. Thus \bar{E} is convex.

(c) Let Y be a subspace of X . If $Y = X$, then $Y^\circ = X \neq \emptyset$. Conversely, let $Y^\circ \neq \emptyset$. If $a \in Y^\circ$, then $\exists r > 0$ such that $U(a, r) \subset Y$. Now, for every $x \neq 0$ in X ,

$$a + r \frac{x}{2\|x\|} \in U(a, r) \subset Y \text{ so that } x \in Y. \text{ Thus } Y = X.$$

$$\left(\because \|a - (a + r \frac{x}{2\|x\|})\| \leq \frac{r}{2} \frac{\|x\|}{\|x\|} = \frac{r}{2} < r. \right)$$

- (d) Let $f: X \rightarrow K$ be linear and $f \neq 0$. Let $a \in X$ with $f(a) = 1$. If E is open in X and $x \in E$, then there exists $r > 0$ such that $U(x, r) \subset E$. For $k \in K$ with $|k| < \frac{r}{\|a\|}$ we have
- ~~$x + ka \in U(x, r) \subset E$~~
- so that $f(x) + k \in f(E)$, that is, $f(E)$ is open.
- used
- $\Rightarrow \|x - (x + ka)\| = \|k\| \|a\| < r$.
- The theorem is proved.

Theorem 2. Let Y be a subspace of a nls X . Then Y and X/Y are nls with the norm induced from X .

- (b) Let Y be a closed subspace of a nls X . We define a quotient space X/Y with the norm given by

$$\|x+Y\| = \inf \{ \|x+y\| ; y \in Y \}$$

for $x+Y \in X/Y$. Then the quotient space X/Y with the above quotient norm is a nls. A sequence $\{x_n+Y\}$ in X/Y converges to $x+Y$ in X/Y iff there exists a sequence $\{y_n\}$ in Y such that $x_n+y_n \rightarrow x$ in X .

- (c) If X_1, X_2, \dots, X_n are nls with norms $\|\cdot\|_1, \|\cdot\|_2, \dots, \|\cdot\|_n$ respectively, then for $1 \leq p < \infty$, the product space

$$X = X_1 \times X_2 \times \dots \times X_n \text{ with } \|x\|_p = \begin{cases} \left(\sum_{j=1}^n \|x(j)\|^p \right)^{1/p}, & \text{if } 1 \leq p < \infty \\ \max_{1 \leq j \leq n} \|x(j)\|_j, & \text{if } p = \infty \end{cases}$$

for $x = (x(1), \dots, x(n)) \in X$ is a nls. A sequence $\{x_n\}$ converges to $x = (x(1), \dots, x(n)) \in X$ iff the sequence $(x_n(j))$ converges to $x(j)$ for each $j = 1, 2, \dots, n$.

Proof(a) Obviously, \mathbb{Y} is a nls. Next, if $x, y \in \mathbb{Y}$, then there exist sequences $\{x_n\}$ and $\{y_n\}$ in \mathbb{Y} such that $x_n \rightarrow x$ and $y_n \rightarrow y$. Then for $k, k' \in K$, $kx_n + k'y_n \in \mathbb{Y}$ and tends to $kx + k'y$. Thus $kx + k'y \in \mathbb{Y}$ which shows that \mathbb{Y} is a nls.

(b) Clearly, $\|x + \mathbb{Y}\| \geq 0$ for every $x \in X$. If $\|x + \mathbb{Y}\| = 0$, then there is a sequence $\{y_n\} \in \mathbb{Y}$ such that $x + y_n \rightarrow 0$. Since \mathbb{Y} is closed and $y_n \rightarrow -x$, we see that $-x \in \mathbb{Y}$, or $x + \mathbb{Y} = 0$. Further, for $x, z \in X$,

$$\begin{aligned}\| (x+z) + \mathbb{Y} \| &= \inf \{ \|x+z+y\|; y \in \mathbb{Y} \} \\ &= \inf \{ \|x+y_1 + z + y_2\|; y_1, y_2 \in \mathbb{Y} \} \\ &\leq \inf \{ \|x+y_1\|; y_1 \in \mathbb{Y} \} + \inf \{ \|z+y_2\|; y_2 \in \mathbb{Y} \} \\ &= \|x + \mathbb{Y}\| + \|z + \mathbb{Y}\|.\end{aligned}$$

Similarly, $\|kx + \mathbb{Y}\| = |k| \cdot \|x + \mathbb{Y}\|$ for $k \in K$ and $x \in \mathbb{Y}$. This proves that $\|\cdot\|$ is a norm in X/\mathbb{Y} .

Now, let $\{x_n + \mathbb{Y}\}$ be a sequence in X/\mathbb{Y} . If $x_n + \mathbb{Y} \rightarrow x + \mathbb{Y}$ in X/\mathbb{Y} , then for every m , there exists n_m such that for $n \geq n_m$, $\| (x_n + \mathbb{Y}) - (x + \mathbb{Y}) \| < \gamma_m$.

Set $n_1 < n_2 < \dots$. Since

$\| (x_n + \mathbb{Y}) - (x + \mathbb{Y}) \| = \inf \{ \|x_n - x + y\|; y \in \mathbb{Y} \}$, choose for $n_m \leq n < n_{m+1}$, $y_n \in \mathbb{Y}$ such that $\|x_n - x + y_n\| < \gamma_m$, $m = 1, 2, \dots$. Then it follows that $x_n + y_n \rightarrow x$ in X .

Conversely, let $\{y_n\}$ be a sequence in \mathbb{Y} such that $x_n + y_n \rightarrow x$ in X . Then

$\| (x_n + \mathbb{Y}) - (x + \mathbb{Y}) \| = \| (x_n - x) + \mathbb{Y} \| \leq \|x_n - x + y_n\|$ for every n . Hence $x_n + \mathbb{Y} \rightarrow x + \mathbb{Y}$ in X/\mathbb{Y} .

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(c) For $1 \leq p \leq \infty$, it is clear that $\|x\|_p > 0$, $\|x\|_p = 0$ iff $x=0$, and $\|\kappa x\|_p = |\kappa| \|x\|_p$ for $x \in X$ and $\kappa \in K$. For $p=1$ and $p=\infty$, it is also clear that $\|x+y\|_p \leq \|x\|_p + \|y\|_p$.

for $x, y \in X$. For $1 < p < \infty$, the above inequality follows from Minkowski's inequality, we obtain

$$\begin{aligned}\|x+y\|_p &\leq \left(\sum_{j=1}^n (\|x(j)\|_p + \|y(j)\|_p)^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{j=1}^n \|x(j)\|_p^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^n \|y(j)\|_p^p \right)^{\frac{1}{p}} \\ &= \|x\|_p + \|y\|_p\end{aligned}$$

for $x = (x(1), x(2), \dots, x(n))$, $y = (y(1), y(2), \dots, y(n))$.

Thus, $\|\cdot\|_p$ is a norm on X for $1 \leq p \leq \infty$.

Finally, it can be checked that $x_n \rightarrow x$ iff $x_n(j) \rightarrow x(j)$ in X_j for each $j = 1, 2, \dots, n$.

The theorem is proved.

Four Major Examples on norms:

As we go on considering various properties of a norm in the later stages, each of the four examples (we give here) will be checked for the properties.

(A) The spaces K^n . On K , the absolute value $\|\cdot\|$ defines a norm, called the normal norm. On the product linear space K^n , there exist a variety of norms. Some of them are given by the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$, as considered at the beginning of the page ^{see} (c), where we let $\|x(j)\|_p = |x(j)|$ for $j = 1, 2, \dots, n$, $x(j) \in K$. Thus, for $x = (x(1), \dots, x(n)) \in K^n$,

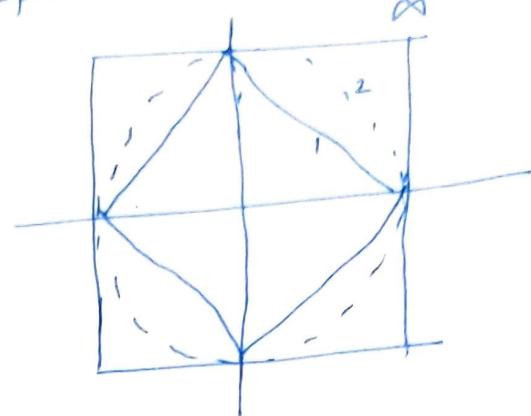
$$\|x\|_p = \begin{cases} (|x(1)|^p + \dots + |x(n)|^p)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \max \{|x(1)|, \dots, |x(n)|\} & \text{if } p = \infty. \end{cases}$$

For $K = \mathbb{R}$ and $n = 2$, the unit spheres

$$\{x \in \mathbb{R}^2, \|x\|_p = 1\}$$

for $p = 1, 2$ and ∞ are shown in

$\|\cdot\|_2 \rightarrow$ sometimes called
Euclidean norm on K .



(B) Sequences spaces.

Let $1 \leq p \leq \infty$. Consider $X = \ell^p$, the space of all p -summable scalar sequences if $1 \leq p < \infty$ and the space of all bounded scalar sequences if $p = \infty$.

For $x = (x(1), x(2), \dots) \in \ell^p$, let

$$\|x\|_p = \begin{cases} \left(\sum_{j=1}^{\infty} |x(j)|^p \right)^{1/p}, & \text{if } 1 \leq p < \infty \\ \sup_{j=1,2,\dots} |x(j)|, & \text{if } p = \infty. \end{cases}$$

We, already seen that $d_p(x, y) = \|x - y\|_p$ is a metric on ℓ^1 . Thus, $\|\cdot\|_p$ is a norm on ℓ^p .

It is easy to prove that if $1 \leq p < \infty$ and $x \in \ell^p$, then $\|x\|_\infty \leq \|x\|_p$ so that $\ell^p \subseteq \ell^\infty$. Also, if $1 \leq p \leq p' < \infty$, then

$$\|x\|_{p'} = \left(\sum_{n=1}^{\infty} |x(n)|^{p'} \right)^{1/p'} \leq \underbrace{\left(\sum_{j=1}^{\infty} |x(j)|^p \right)^{1/p}}_{\text{Jensen's inequality.}} = \|x\|_p.$$

These can be proved as follows:

We can assume w.l.o.g. that $\sum_{j=1}^{\infty} |x(j)|^p = 1$. Then $|x(j)| \leq 1$ for all $j = 1, 2, \dots$. Since $p \leq p'$, it follows that

$$|x(j)|^{p'} \leq |x(j)|^p \text{ for all } j = 1, 2, \dots \text{ Hence}$$

$$\sum_{j=1}^{\infty} |x(j)|^{p'} \leq \sum_{j=1}^{\infty} |x(j)|^p = 1.$$

Taking p th roots, we have $\|x\|_{p'} \leq 1 = \|x\|_p$.

Thus, we see that $\ell^p \subset \ell^{p'}$ if $1 \leq p \leq p' \leq \infty$.

(C) L^p -spaces : Intentionally leaving them. We shall discuss later.

(D) Function Spaces.

Let T be a set and $B(T)$ be the set of all bounded k -valued functions on T . For $x \in B(T)$, let $\|x\|_\infty = \sup \{ |x(t)| ; t \in T \}$ here x is a mapping.

Then $\|\cdot\|_\infty$ is a norm on $B(T)$, called the sup norm.

If T is a metric space, consider the following subspaces:

$$C(T) = \{x \in B(T); x \text{ is continuous on } T\}$$

$$C_0(T) = \{x \in C(T); \text{ for every } \epsilon > 0, \exists \text{ a compact set } E \subset T \text{ such that } |x(t)| < \epsilon \text{ for } t \notin E\}$$

$$C_c(T) = \{x \in C_0(T); \exists \text{ a compact set } E \subset T \text{ such that } x(t) = 0 \text{ for } t \notin E\}.$$

The elements of $C_0(T)$ are called continuous functions vanishing at infinity, and those of $C_c(T)$ are called the continuous functions with compact support. In case T is compact, $C(T) = C_0(T) = C_c(T)$.

If $T = [a, b]$, then for $n = 1, 2, \dots$, we define

$$C^n([a, b]) = \{x \in C([a, b]); x \text{ is } n\text{-times differentiable and the } n\text{-th derivative } x^{(n)} \in C([a, b])\}$$

For $x \in C^n([a, b])$, $\|x\| = \|x\|_\infty + \|x'\|_\infty + \dots + \|x^{(n)}\|_\infty$ defines a norm.

(E) Inner product spaces : We shall see after Banach space.