

Continuity of Linear Maps.

We deal with linear maps between two \mathbb{K} -spaces.

Theorem: Let X and Y be \mathbb{K} -spaces and $f: X \rightarrow Y$ a linear map. Then the following conditions are equivalent:

- (i) f is bounded on some closed ball about 0 of positive radius.
 - (ii) f is continuous at 0 .
 - (iii) f is continuous at every point of X .
 - (iv) f is uniformly continuous on X .
 - (v) $\|f(x)\| \leq \alpha \|x\|$ for every $x \in X$ and some $\alpha \in \mathbb{R}$.
- If in particular, $Y = K$, then each of the following above conditions/statements is equivalent to
- (vi) the hyperspace $Z(f)$ is closed in X ($f \neq 0$).

Def: Let X be a linear space over K .

(i) A linear map $f: X \rightarrow K$ is called a linear functional on X .

(ii) A subset H of X is called a hyperspace in X if H is a maximal proper subspace of X . If H is a hyperspace in X and $a \in X$, then $a + H$ is called a hyperplane in X .

$$Z(f) = \{x \in X; f(x) = 0\} \rightarrow \text{null space of } f.$$

Proof of the Theorem.

(i) \Rightarrow (v) Let $\|f(x)\| \leq \beta$, whenever $\|x\| \leq r$ for some positive numbers $\beta, r \in \mathbb{R}$. Then for $x \neq 0$,

$$\left\| \frac{rx}{\|x\|} \right\| \leq r$$

so that $\|f(x)\| = \|x\| \cdot f\left(\frac{rx}{\|x\|}\right) \cdot \frac{1}{r} \leq \beta \cdot \frac{\|x\|}{r} = \alpha \|x\|$,

where $\alpha = \frac{\beta}{r}$.

Hence (v) follows.

$\rightarrow \|x - 0\| \leq r$
closed ball about

by this is possible since f is a linear map, $r, \|x\|$ are constants.

Note: x is not equal to 0

Continuity of Linear Maps.

We deal with linear maps between two ~~sets~~ spaces.

Theorem: Let X and Y be sets and $f: X \rightarrow Y$ a linear map. Then the following conditions are equivalent:

- (i) f is bounded on some closed ball about 0 of positive radius.
- (ii) f is continuous at 0 .
- (iii) f is continuous ~~at~~ on X .
- (iv) f is uniformly continuous on X .
- (v) $\|f(x)\| \leq \alpha \|x\|$ for every $x \in X$ and some $\alpha \in \mathbb{R}$.
If in particular, $Y = K$, then each of the following above conditions/statements is equivalent to
- (vi) the hyperspace $Z(f)$ is closed in X ($f \neq 0$).

Def: Let X be a linear space over K .

- (i) A linear map $f: X \rightarrow K$ is called a linear functional on X .
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Proof of the Theorem.

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$$\left\| \frac{rx}{\|x\|} \right\| \leq r$$

so that $\|f(x)\| = \underbrace{\|x\| \cdot f\left(\frac{rx}{\|x\|}\right)}_{\substack{\rightarrow \text{closed ball about } 0 \\ \|x\|}} \leq \beta \cdot \frac{\|x\|}{r} = \alpha \|x\|$,

$$\text{where } \alpha = \frac{\beta}{r}.$$

Hence (v) follows.

\hookrightarrow This is possible since f is a linear map, $r, \|x\|$ are constants.