

Completeness of a Norm

A nls X (over K) is called a Banach space if X is complete in the metric induced by the norm.

We give a characterization of Banach spaces among nls in terms of convergent series:

Let $\{x_n\}$ be a sequence in a nls X and

$$S_m = \sum_{n=1}^m x_n, \quad m=1,2,3,\dots$$

The series $\sum_{n=1}^{\infty} x_n$ is said to be convergent in X if the sequence of partial sums $\{S_m\}$ converges in X ; if $S_m \rightarrow x \in X$, we denote/write $x = \sum_{n=1}^{\infty} x_n$ and call it the sum of the series. A series $\sum_{n=1}^{\infty} x_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Theorem: A nls X is a Banach space iff every absolutely convergent series of elements in X is convergent in X .

Proof Let X be a Banach space. Let $x_n \in X, n=1,2,\dots$, with

$$\sum_{n=1}^{\infty} \|x_n\| < \infty. \quad \text{If } S_m = \sum_{n=1}^m x_n, \quad m=1,2,\dots, \text{ then for } m, j=1,2,\dots,$$

$\|S_{m+j} - S_m\| \leq \sum_{n=m+1}^{m+j} \|x_n\|$. This shows the sequence $\{S_m\}$ is Cauchy in X . Hence $\{S_m\}$ converges in X , i.e., $\sum_{n=1}^{\infty} x_n$ is convergent in X .

Conversely, assume that every absolutely convergent series is convergent in X . We show that X is a Banach space. Let $\{S_m\}$ be a Cauchy sequence in X . Let m_1 be an integer such that $\|S_m - S_{m_1}\| \leq 1$ for all $m \geq m_1$. Inductively, define m_2, m_3, \dots such that $\frac{m_n}{m_{n+1}} \leq \frac{1}{2}$ and $\|S_m - S_{m_n}\| \leq \frac{1}{2^n}$ for $n=1,2,\dots$ and for all $m \geq m_n$.

Let $x_n = s_{m_{n+1}} - s_{m_n}$. Then it follows that

$$\sum_{n=1}^{\infty} \|x_n\| \leq \sum_{n=1}^{\infty} \epsilon_{n/2} < \infty.$$

By assumption, $\sum_{n=1}^{\infty} x_n$ is convergent in X . Since

$$s_{m_n} = s_{m_1} + \sum_{j=1}^{n-1} x_j, \quad n=1, 2, \dots,$$

$\{s_{m_n}\}$ converges in X . Thus, the Cauchy sequence $\{s_n\}$ has a convergent subsequence in X , and hence itself converges in X . Hence X is Banach.

Theorem: A Banach space cannot have a denumerable basis.

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1-1 correspondance with naturals.

Definition.

Let X be a nls. A set $\{x_n \in X; \|x_n\|=1, n=1, 2, \dots\}$ is called a Schauder basis for X if ^{for} every $x \in X$, there exist unique $k_n \in \mathbb{K}$, $n=1, 2, \dots$, such that

$$x = \sum_{n=1}^{\infty} k_n x_n.$$

Theorem: (a) Let X be a nls. and Y a closed subspace of X . Then Y with the induced norm and X/Y with the quotient norm are Banach iff X is Banach.

(b) Let X_1, X_2, \dots, X_n be nls, and $X = X_1 \times X_2 \times \dots \times X_n$, the product space with any of the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$. Then X_1, X_2, \dots, X_n are Banach spaces iff X is Banach.

Sup
H.W.

Theorem: Let $X = \{0\}$ and Y be a n.l.s. Then $BLL(X, Y)$ is Banach iff Y is Banach. In particular, X' is Banach for every n.l.s. X .

Theorem: Let X be a n.l.s. and Y a Banach space.

- (a) Let X_n be a dense subspace of X and $F_n \in BLL(X_n, Y)$. Then there exists a unique $F \in BLL(X, Y)$ such that $F|_{X_n} = F_n$. Further, $\|F\| = \|F_n\|$.
- (b) Let E be a subset of X such that $\text{span}\{E\}$ is dense in X , and $F_n \in BLL(X, Y)$ such that $\|F_n\| \leq \alpha$ for some $\alpha > 0$ and $n = 1, 2, \dots$. If $\{F_n(x)\}$ converges for every $x \in E$, then for some $F \in BLL(X, Y)$ $F_n(x) \rightarrow F(x)$ for every $x \in X$.

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