

Lemma Let X be a nls. over \mathbb{R} , E an open convex subset of X and Y a subspace of X such that $E \cap Y = \emptyset$. If Y is not a hyperplane in X , then there exists $x \in X$ such that $x \notin Y$ and $E \cap \text{span}\{Y, x\} = \emptyset$.



Thm B

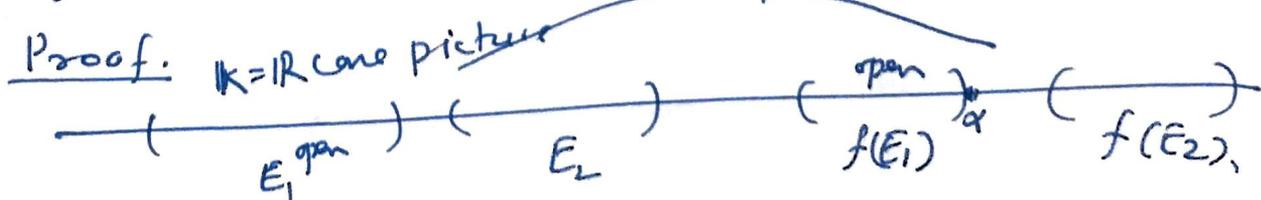
Theorem Let X be a nls over \mathbb{R} , E a nonempty open convex subset of X , and Y a subspace of X such that $E \cap Y = \emptyset$. Then there exists a closed hyperplane H in X such that $Y \subset H$ and $E \cap H = \emptyset$. In other words, there exists $f \in X'$ such that $f = 0$ on Y , but $f(x) \neq 0$ for every $x \in E$.

Hahn-Banach Separation Theorem:

Let X be a nls over K and E_1, E_2 nonempty disjoint convex subsets of X with E_1 open. Then there exists $f \in X'$ and $\alpha \in \mathbb{R}$ such that

$$\text{Re } f(x_1) < \alpha \leq \text{Re } f(x_2)$$

for all $x_1 \in E_1$ and $x_2 \in E_2$.



First, let $K = \mathbb{R}$. The set $E_1 - E_2$ is nonempty, open and convex. Also, E_1 and E_2 are disjoint, $0 \notin E_1 - E_2$. So by previous theorem, with $Y = \{0\}$ and $E_1 - E_2$ in place of E , there exists $f \in X'$ such that $0 \notin f(E_1 - E_2)$, i.e., $f(E_1) \cap f(E_2) = \emptyset$.

Since E_1 is open and f is a nonzero linear functional, then $f(E_1)$ is open in \mathbb{R} . Since $f(E_1)$ and $f(E_2)$ are disjoint convex subsets of \mathbb{R} , they are non-overlapping intervals in \mathbb{R} . We can assume, by multiplying f by -1 if necessary, that $f(E_1)$ lies to the left of $f(E_2)$. Since $f(E_1)$ is an open interval, we can let α to be the right endpoint of $f(E_1)$, and obtain

$$f(x_1) < \alpha \leq f(x_2) \quad \forall x_1 \in E_1, \forall x_2 \in E_2.$$

Next, let $K = \mathbb{C}$. Regard X as a linear functional over \mathbb{R} . Then by the case $X = \mathbb{R}$, there exists a continuous real-linear functional u on X and some $\alpha \in \mathbb{R}$ such that

$$u(x_1) < \alpha \leq u(x_2) \quad \forall x_1 \in E_1, \forall x_2 \in E_2.$$

For $x \in X$, let $f(x) = u(x) - i u(ix)$. Then by the previous theorem, f is a continuous complex-linear functional on X and $\operatorname{Re} f = u$. Thus f meets the requirements.

~~The Hahn-Banach~~

Corollary: Let X be a nls over K and E be a convex subset of X with nonempty interior. If $b \in E^\circ$, then there exists a nonzero $f \in X'$ such that

$$\operatorname{Re} f(x) \leq \operatorname{Re} f(b) \quad \forall x \in E.$$

Proof: Since E° is nonempty, open, convex, set $E_1 = E^\circ$ and $E_2 = \{b\}$ in HBS, and fix $f \in X'$ such that

$$\operatorname{Re} f(x) < \operatorname{Re} f(b) \quad \forall x \in E^\circ.$$

Fix $a \in E^\circ$. Then $\exists \epsilon > 0$ such that $U(a, \epsilon) \subset E$. If $x \in E_2$, it follows that for $0 < r < 1$, $U(ra + (1-r)x, r) \subset E$. Thus, $ra + (1-r)x \in E^\circ \quad \forall x \in E$ and $0 < r < 1$. Letting $r = 1/n$ and $n \rightarrow \infty$, we obtain by the continuity of f

that $\operatorname{Re} f(x) \leq \operatorname{Re} f(b)$.

The Corollary is proved.

Hahn-Banach Extension Theorem

Let X be a nls over K and Y a subspace of X . If $g \in Y'$, then $\exists f \in X'$ such that $f|_Y = g$ and $\|f\| = \|g\|$.

Proof: Let, first, $K = \mathbb{R}$. If $g \equiv 0$ on Y , let $f \equiv 0$ on X . If $g \neq 0$ on Y , let $a \in Y$ with $g(a) = 1$, and consider $E = U(a, \frac{1}{\|g\|})$. If $y \in E \cap Y$, then $|g(y) - 0| = |g(y-a)| \leq \|g\| \cdot \|y-a\| < 1$ so that $g(y) \neq 0$. Thus, $E \cap Z(g) = \emptyset$. By Thm B, with $Z(g)$ in place of Y , $\exists f \in X'$ such that $f \equiv 0$ on $Z(g)$ and $f(x) \neq 0 \forall x \in E$. Since $a \in E$, we can assume that $f(a) = 1$. Then $f|_Y$ and g are linear functionals on Y such that $f|_Y(a) = 1 = g(a)$ and $Z(g) \subset Z(f|_Y)$. Hence $\underline{f|_Y = g}$.

Finally, if $x \notin Z(f)$, then $a - \frac{x}{f(x)} \in Z(f)$, so that $a - \frac{x}{f(x)} \in E$, i.e., $\|a - (a - \frac{x}{f(x)})\| = \|\frac{x}{f(x)}\| \geq \frac{1}{\|g\|}$, or

$$|f(x)| \leq \|g\| \cdot \|x\|.$$

Thus, $\|f\| \leq \|g\|$. Since the reverse inequality is obvious, then we see that $\|f\| = \|g\|$.

Now, let $K = \mathbb{C}$. Regard X as a linear space over \mathbb{R} . By the case $K = \mathbb{R}$, \exists a real linear functional u on X such that $u|_Y = \operatorname{Re} g$ and $\|u\| = \|\operatorname{Re} g\|$. Let $f = u(x) - i v(x) \forall x \in X$. Then by Lemma A, we have $f|_Y = g$ and $\|f\| = \|u\|$.

$$\|f\| = \|u\| = \|\operatorname{Re} g\| = \|g\|.$$

The theorem is proved.

Corollary: Let X be a nls. over K and $0 \neq a \in X$. Then there exists $f \in X'$ with $f(a) = \|a\|$ and $\|f\| = 1$.

Banach space (we shall see next period).

Refer
p. 114
we discuss
about it.