

- 13 -

Bounded Linear Maps.

A linear map from a nls X to a nls Y is continuous iff it maps bounded sets in X onto bounded sets in Y . For this reason, a continuous linear map is called a bounded linear map.

Note: such maps are not bounded in the usual sense, i.e., there is no $\alpha \in \mathbb{R}$ such that $\|f(x)\| \leq \alpha \forall x \in X$.

In fact, no nonzero linear map can be bounded in the usual sense. ~~can't be bounded~~

→ We shall denote the set of all bounded linear maps from a nls X to a nls Y by $BL(X, Y)$; if $Y = X$, we shall denote it by $BL(X)$ and if $Y = K$, then by X' .

Theorem: Let X and Y be nls. over K . Under pointwise addition and scalar multiplication, $BL(X, Y)$ is a linear space over K . For a linear map f from X to Y , let $\|f\| = \sup \{ \|f(x)\| : x \in X, \|x\| \leq 1 \}$.

Then $\|\cdot\|$ is a norm on $BL(X, Y)$, called the operator norm. Let $f \in BL(X, Y)$. For every $x \in X$,

$$\|f(x)\| \leq \|f\| \cdot \|x\|.$$

In fact, we have

$$\|f\| = \sup \{ \|f(x)\| : x \in X, \|x\| = 1 \}$$

$$= \inf \{ \alpha > 0 : \|f(x)\| \leq \alpha \|x\| \text{ for every } x \in X \},$$

Proof. It is clear that $BL(X, Y)$ is a nls. under pointwise operations of addition and scalar multiplication and the given definition of $\|f\|$.

Let $f \in BL(X, Y)$. If $x = 0$, then $f(x) = 0$ ($\because f$ is linear).

Let $x \in X$ then $\|x/\|x\|\| = 1$ so that

$$\|f(x)\| = \|f(x/\|x\|\|) \cdot \|x\| \leq \|f\| \cdot \|x\|.$$

This also shows that

$$\|f\| = \sup \{ \|f(x)\| : x \in X, \|x\|=1 \}.$$

Finally, if $\alpha > 0$ with $\|f(x)\| \leq \alpha \|x\|$ for every $x \in X$, then for $x \in X$ with $\|x\|=1$, $\|f(x)\| \leq \alpha$ so that $\|f\| \leq \alpha$. Hence $\|f\| = \inf \{ \alpha > 0 : \|f(x)\| \leq \alpha \|x\| \text{ for every } x \in X \}$. The theorem is proved.

Hahn-Banach Theorem:

One of the most fundamental results in functional analysis concerns the extensions of a continuous linear functional $g: Y \rightarrow K$, where Y is a subspace of a normed space X to an $f: X \rightarrow K$ which is also continuous and linear, and in fact, has the same norm as g .

We have proved earlier, g can readily be extended to a linear functional on X . Also, by the uniform continuity of g on Y , we can extend it continuously to the closure \bar{Y} of Y and then to X by Tietze's extension theorem.

But it is far from clear how to obtain an extension of g to X which is both linear and continuous. This will be accomplished by the Hahn-Banach extension theorem.

Lemma. Let X be a nls over \mathbb{C} , and let $X_{\mathbb{R}}$ denote X , regarded as a linear space over \mathbb{R} . If f is a complex-linear functional on X , then ~~Re f~~ $\text{Re } f$ is a real-linear functional on $X_{\mathbb{R}}$ and $\|f\| = \|\text{Re } f\|$. Moreover, for all $x \in X$,

$$f(x) = \text{Re } f(x) - i \text{Im } f(ix).$$

15

Conversely, if u is a real-linear functional on X_R and we let for all $x \in X$

$$f(x) = u(x) - i u(ix),$$

then f is a complex-linear functional on X , $\operatorname{Re} f = u$ and $\|f\| = \|u\|$.

Proof. Let f be a complex-linear functional on X . Then it is clear that $\operatorname{Re} f$ is real-linear on X_R . Also, for all $x \in X$, ~~$f(ix) = i f(x)$~~ so

that

$$\begin{aligned}\operatorname{Re} f(x) - i \operatorname{Re} f(ix) &= \operatorname{Re} f(x) - i \operatorname{Re} i f(x) \\ &= \operatorname{Re} f(x) + i \operatorname{Im} f(x) \\ &= f(x).\end{aligned}$$

Now, $\|f\| = \sup \{ |f(x)| ; x \in X, \|x\| \leq 1 \}$

and $\|\operatorname{Re} f\| = \sup \{ |\operatorname{Re} f(x)| ; x \in X, \|x\| \leq 1 \}$.

Hence $\|\operatorname{Re} f\| \leq \|f\|$.

On the other hand, for $x \in X$ with $\|x\| \leq 1$, find $k \in \mathbb{C}$ with $|k| = 1$ such that $k f(x) = f(kx) = |f(x)|$. Then $\|kx\| \leq 1$ and $\operatorname{Re} f(kx) = f(kx) = |f(x)|$. Hence $\|f\| \leq \|\operatorname{Re} f\|$. Hence $\|\operatorname{Re} f\| = \|f\|$.

If u is a real-linear functional on X_R , then $f(x) = u(x) - i u(ix)$, $x \in X$, is complex-linear since $f(ix) = u(ix) - i u(-x) = i(-i u(ix) + u(x)) = i f(x)$. Clearly, $\operatorname{Re} f = u$, and hence $\|f\| = \|u\|$ by above. The ~~previous~~ lemma is proved.