

Theorem: Two nls. X and Y over K are linearly homeomorphic if and only if there exists a linear map $f: X \rightarrow Y$ which is onto and satisfies

$$\alpha \|x\| \leq \|f(x)\| \leq \beta \|x\| \quad \forall x \in X$$

and some $\alpha, \beta > 0$.

In particular, if X is complete, and Y is linearly homeomorphic to X , then Y is complete.

Proof. If a map $f: X \rightarrow Y$ in the above stipulated properties exists, then it is bijective, and also continuous (by (v) of the first continuous theorem of nls.) The inverse map $f^{-1}: Y \rightarrow X$ is also linear and satisfies

$$\|f^{-1}(y)\| \leq \frac{1}{\alpha} \|y\| \quad \forall y \in Y.$$

Hence f is a homeomorphism onto (by (v) of the first continuous theorem of nls.)

Conversely, if $f: X \rightarrow Y$ is a linear homeomorphism onto, then by (v) of the first cont. thm. of nls., we have $\|f(x)\| \leq \beta \|x\|$ and $\|f^{-1}(y)\| \leq r \|y\|$ for all $x \in X$, $y \in Y$ and some $\beta, r \in \mathbb{R}$. Then, the result follows by setting $r = \frac{1}{\alpha}$. The theorem is proved.

Corollary:

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a linear space X define equivalent metrics if and only if there exist $\alpha, \beta > 0$ such that $\alpha \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1$, for all $x \in X$. Such norms are called equivalent norms.

Idea of the proof (H-W)

Take f to be the identity map in the above theorem.

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Idea of the proof. (H-W)

Take f to be the identity map in the above theorem.

Theorem Let X be a nls over K and \mathcal{Y} a subspace of X of finite dimension m . Then every bijective linear map $f: K^m \rightarrow \mathcal{Y}$ is also a homeomorphism, and \mathcal{Y} is closed in X , where K^m is given any of the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$.

Corollary: (a) If X and \mathcal{Y} are nls. of dimension m over K , then they are linearly homeomorphic. In particular, all norms on K^m are equivalent.
(b) If X is a finite dimensional nls., then X is complete.

Lemma (F. Riesz, 1918).

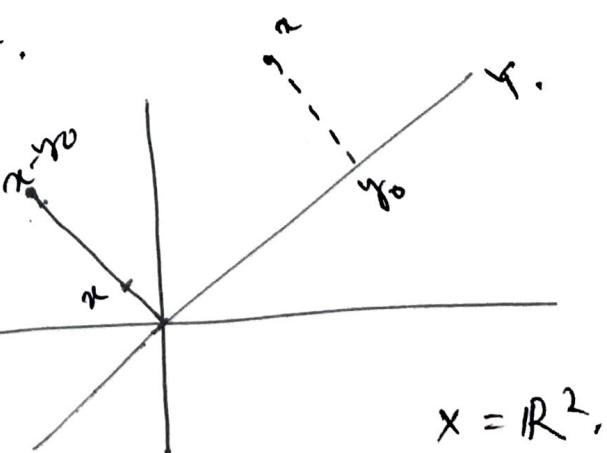
Let X be a nls. and \mathcal{Y} a closed subspace of X with $\mathcal{Y} \neq X$. Let $0 < r < 1$ be a real number. Then there exists $x_r \in X$ such that $\|x_r\| = 1$, and $r \leq d(x_r, \mathcal{Y}) \leq 1$.

Proof. Let X be a nls., and consider $x \in X$ with $x \notin \mathcal{Y}$. Let $d = d(x, \mathcal{Y})$. Since \mathcal{Y} is closed, $d > 0$. Again, since $r \leq 1$, there exists $y_0 \in \mathcal{Y}$ such that $\|x - y_0\| \leq d/r$.

$$\text{Let } x_r = \frac{x - y_0}{\|x - y_0\|}.$$

Then $\|x_r\| = 1$, and

$$\|x - x_r\| y + y_0 \in \mathcal{Y} \text{ if } y \in \mathcal{Y}.$$



Hence

$$\|y - x_r\| = \left\| y - \frac{x - y_0}{\|x - y_0\|} \right\|$$

$$= \frac{\|(\|x - y_0\| y + y_0) - x\|}{\|x - y_0\|} \geq \frac{d}{\|x - y_0\|} > r,$$

which is true for every $y \in \mathcal{Y}$, $d(x_r, y) = \inf \{ \|y - x_r\|; y \in \mathcal{Y} \} \geq r$.

Second Proof. Since Y is a closed subspace of X and $Y \neq X$, the quotient ~~space~~ space $X/Y \neq \{0\}$. Let $z \in X$ be such that $\|z+Y\|=1$. Now,

$$\|z+Y\| = \inf \{\|z+y\|; y \in Y\}.$$

Since $r < 1$, there is $y_r \in Y$ such that

$$1 \leq \|z+y_r\| \leq r.$$

Set $x_r = \frac{z+y_r}{\|z+y_r\|}$. Then $\|x_r\|=1$ and

$$\begin{aligned} d(x_r, Y) &= \frac{1}{\|z+y_r\|} \cdot d(z+y_r, Y) = \frac{1}{\|z+y_r\|} d(z, Y) \\ &= \frac{\|z+Y\|}{\|z+y_r\|} = \frac{1}{\|z+y_r\|} \geq r. \end{aligned}$$

Since $0 \in Y$, then we see that $d(x_r, Y) \leq 1$. (done!)

Remark The above lemma says that if Y is a closed proper (i.e. $\neq X$) subspace of a nls X , then there exist points on the unit sphere of X whose distances from Y are as close as to 1 (as we please). There may not be a point, though, whose distance is exactly 1. (we shall see the last part later.)

Theorem

Let X be a nls. Then the closed unit ball in X is compact iff X is finite dimensional.

Thus, the Heine-Borel Theorem holds in X iff X is finite dimensional. \downarrow

Let X be a complete metric space and $E \subset X$. E is compact iff E is closed and totally bounded. In particular, if $X = \mathbb{K}^n$ and $d = d_n$, then E is compact iff E is closed and totally bounded.