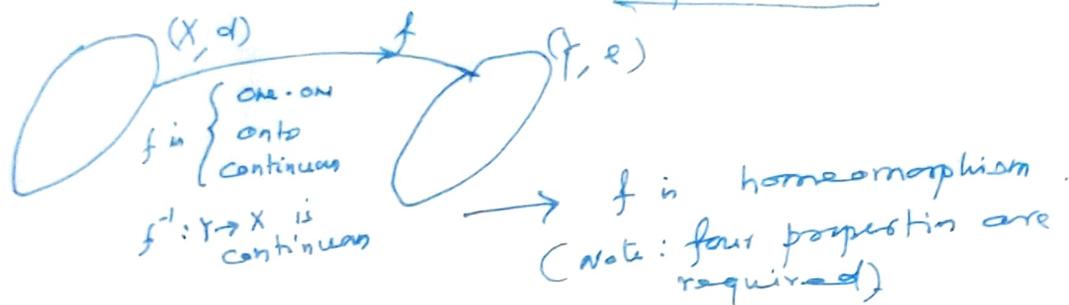


Unlike in the case of linear spaces, and linear maps, a function $f: X \rightarrow Y$, where X and Y are metric spaces, may be continuous and one to one without $f^{-1}(E)$ or $f^{-1}: R(f) \rightarrow X$ being continuous.

Ex: $X = [0, 2\pi)$, $Y = \mathbb{C}$, $f(t) = e^{it}$

Definition Let X and Y be metric spaces. If $f: X \rightarrow Y$ is one to one, onto and continuous with $f^{-1}: Y \rightarrow X$ is also continuous, then f is called a homeomorphism and X and Y are said to be homeomorphic. If $X = Y$, then ~~the~~ the two metrics are said to be equivalent.



\rightarrow We can obtain a very special kind of homeomorphism (we call it an isometry onto): If $f: X \rightarrow Y$ is an onto function such that

$$d(x, x') = e(f(x), f(x')) \quad (\text{distance preserving property})$$

for all $x, x' \in X$. Such a function "preserves" the distance, and is called an isometry onto, and we call ~~the~~ the metric spaces X and Y are isometric.

Definition: Let X be a metric space and $E \subset X$. For $x \in X$, ~~the~~ set $d(x, E) = \inf \{ d(x, y) ; y \in E \}$;

then $d(x, E)$ is called the distance of a point x from the subset E .

Note: if $x \in E$, then $d(x, E)$ vanishes. Thus, we have a very beautiful theorem:

Uryshon's Lemma: If E_0 and E_1 are disjoint closed subsets of a metric space X , then there exists a continuous function $f: X \rightarrow [0, 1]$ such that $f|_{E_0} = 0$ and $f|_{E_1} = 1$.

Pf: For $x \in X$, $f(x) = \frac{d(x, E_0)}{d(x, E_0) + d(x, E_1)}$. Verify. (Homework).

Tietze's Extension Theorem

Let E be a closed subset of a metric space X , and $g: E \rightarrow K$ a continuous function. Then there exists a continuous function $f: X \rightarrow K$ such that $f|_E = g$, and if $|g(x)| \leq \alpha$ for all $x \in E$, then $|f(x)| \leq \alpha$ for all $x \in X$.

~~Proof: A repeated application of Urysohn's Lemma, shows that if $g \in C(E, \mathbb{R})$, $f \in C(X, \mathbb{R})$.~~

Continuous functions on Compact Spaces

Let X be a compact metric space. If $f: X \rightarrow Y$ is continuous, then range of f is a compact subset of Y , i.e. $R(f)$ is a compact subset of Y . Hence

$f: \text{closed subsets of } X \xrightarrow[\text{onto}]{\text{sends}} \text{closed subsets of } Y$.
If, in addition, f is one to one, then $f^{-1}: R(f) \rightarrow X$ is also continuous.

Another property of a continuous function on a compact space is that it is uniformly continuous in the sense that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $d(x, x') < \delta$, we have $d(f(x), f(x')) < \epsilon$; by the continuity of f at $x \in X$, there exists a $\delta_x > 0$ such that $d(x, x') < \delta_x$ implies $d(f(x), f(x')) < \epsilon/2$. Cover X by $U(x, \delta_x/2)$, $x \in X$, and by the compactness of X , we have a finite subcover $U(x_1, \delta_{x_1}/2), \dots, U(x_n, \delta_{x_n}/2)$ of X . Set $\delta = \min\{\delta_{x_1}, \dots, \delta_{x_n}\}/2$, then it works.

Finally, we consider the set $C(X)$ of all K -valued continuous functions on a compact metric space X . For $f, g \in C(X)$, let

$$D(f, g) = \sup\{|f(x) - g(x)|; x \in X\}$$

since $(f-g)(X)$ is compact, it is bounded so that $D(f, g) \in \mathbb{R}$.

Note: D is a metric on $C(X)$, called sup metric on $C(X)$.

A sequence $\{f_n\}$ in $C(X)$ converges to $f \in C(X)$ w.r. to the sup metric iff $\{f_n\}$ converges to f uniformly on X .