

Theorem: Let X be a metric space. Then the following conditions are equivalent:

(i) X is compact;

(ii) Every sequence in X has a convergent subsequence;

(iii) X is complete and totally bounded.

Proof: (i) \Rightarrow (ii). Let X be compact, and $\{x_n\}$ a sequence in X . If it has no convergent subsequence, then for each $y \in X$, there exist $r_y > 0$ and a positive integer n_y such that for $n \geq n_y$, $x_n \notin U(y, r_y)$. Now, ~~$\exists U$~~ $x = U(U(y, r_y); y \in X)$, and since it is compact, there exists $y_1, y_2, \dots, y_n \in X$ such that $x = \bigcup_{j=1}^n U(y_j, r_{y_j})$. Hence, if $n \geq n_j$ for $j=1, 2, \dots, n$, then x_n would not belong to X , a contradiction.

(ii) \Rightarrow (iii). If (ii) holds, then X is complete since every sequence which has a convergent subsequence itself converges.

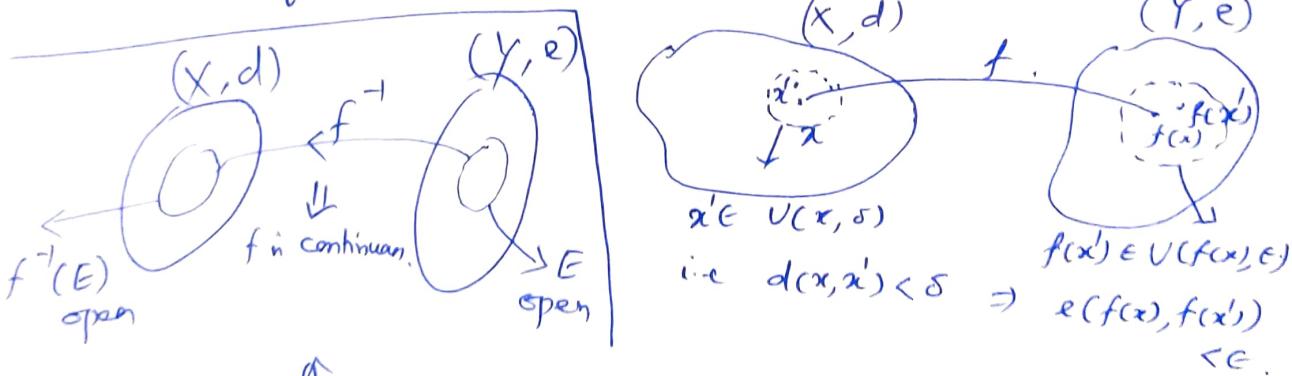
Now, if X is not totally bounded, then there exists an $\epsilon > 0$ such that X cannot be covered by finitely many open balls of radiuses ϵ . Let $x_i \in X$. Having chosen $x_1, x_2, \dots, x_{n-1} \in X$, choose $x_n \in \bigcup_{j=1}^{n-1} U(x_j, \epsilon)$. Then the sequence $\{x_n\}$ in X has no convergent subsequence since any two numbers of it are at a distance greater than or equal to ϵ , which contradicts (ii).

(iii) \Rightarrow (i) proof is easy (Home Work).

- Corollary: Let X be a complete metric space and $E \subset X$.
- (a) E is compact iff E is closed and totally bounded.
 In particular, if $X = K^n$, $d = d_n$, then E is compact iff E is closed and bounded. (Heine-Borel theorem).
- (b) \bar{E} is compact iff E is totally bounded.

Continuity of Functions:

Roughly speaking, a function is continuous if it sends "nearby" points to "nearby" points. More accurately, let X and Y be metric spaces with metric d and e respectively. A function $f: X \rightarrow Y$ is said to be continuous at $x \in X$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $e(f(x), f(x')) < \epsilon$, whenever $d(x, x') < \delta$. f is said to be continuous on X if it is continuous at every $x \in X$.



It is easy to see that f is continuous on X iff $f^{-1}(E)$ is open in X whenever E is open in Y .

* In term of sequences, $f: X \rightarrow Y$ is continuous iff $f(x_n) \rightarrow f(x_0)$ in Y whenever $x_n \rightarrow x$ in X .

