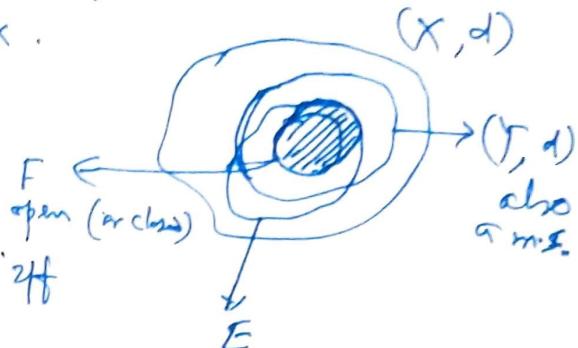


-y -

Finally, if $Y \subset X$, then a metric d on X induces a metric on Y . A set $F \subset Y$ is open (resp. closed) in Y w.r.t. this metric iff $F = E \cap Y$, where ~~$E \subset X$~~ E is open (resp. closed) w.r.t. X .



→ A set E in a metric space

is called dense in X iff $\overline{E} = X$. A metric space X is called separable, if it contains a dense set which is countable.

It should be noted that if X is a separable metric space, and $Y \subset X$, then Y is separable in the induced metric: Let $\{x_1, x_2, \dots\}$ be a dense subset of X . For positive integers n and m , consider $U_{n,m} := U(x_n, r_m)$. If $Y \cap U_{n,m} \neq \emptyset$, then choose $y_{n,m}$ in it.

Sequences.

→ We shall characterize the closed sets in a metric space X in terms of sequences in X .

→ A sequence $\{x_n\}$ in X is a function from the set of all natural numbers to X , its value at n is denoted by x_n .

→ It is said to converge in X if there exists $z \in X$ such that for every $\epsilon > 0$, there is n_0 with ~~$x_n \in U(z, \epsilon)$~~ $x_n \in U(z, \epsilon)$ for $n \geq n_0$.

(Hint: it looks like $|x_n - z| < \epsilon$ for $n \geq n_0$

recall the definition of convergence of a sequence in Real Analysis)

-> -

→ It is clear that there exists at most one such x and when such x exists, we say that $\{x_n\}$ converges to x in X , and we write $x_n \rightarrow x$, we call x the limit of $\{x_n\}$.

(Later we shall see different notations/forms of definition, ...).

You can find a number of examples, skipping them.

→ Quite often, a sequence $\{x_n\}$ in X may not converge in X but the points x_n would get close to each other as we go far out in the sequence.

i.e. $|x_n - x_m| < \epsilon$ for $n, m \geq N_0$. → norm form.

sucha " $[d(x_n, x_m) < \epsilon \text{ for } n, m \geq N_0]$ → metric form.

Such a sequence is called a Cauchy Sequence:

if for every $\epsilon > 0$, $\exists N_0$ such that for all $m, n \geq N_0$, we have $d(x_n, x_m) < \epsilon$.

It is clear that every convergent sequence

is Cauchy and every Cauchy sequence is bounded, i.e. it lies in some big ball in X .

 $d(x_n, x) < \frac{\epsilon}{2}$ for $n \geq N_0$.

\Rightarrow for $n, m \geq N_0$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x, x_m) \\ &= d(x_n, x) + d(x_m, x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

There is a simple useful test to check whether a Cauchy sequence converges: if a ~~subsequence~~ subsequence $\{x_{n_m}\}$ ($n_1 < n_2 < \dots \rightarrow \infty$) of Cauchy sequence $\{x_n\}$ converges, then $\{x_n\}$ itself converges.

Ques. \rightarrow A metric space X is said to be complete if every Cauchy sequence in X converges in X ; i.e., loosely speaking, if every sequence in X which tries to converge is successful in finding some $x \in X$ to converge to -

- An example of a complete metric space is \mathbb{R} with the usual metric.

$$\hookrightarrow \text{recall } d(x,y) = \begin{cases} |x-y| & \text{if } x \neq y \\ 0 & \text{if } x=y. \end{cases}$$

Let $\{x_n\}$ be a Cauchy sequence in \mathbb{R} and consider $\limsup \{x_n\}$, since $\{x_n\}$ is a Cauchy sequence, it is bounded. Hence $\limsup x_n \in \mathbb{R}$. Since every bounded sequence has a convergent subsequence, then the Cauchy sequence $\{x_n\}$ has a subsequence which also converges to $\limsup x_n$. Since $\{x_n\}$ is Cauchy, then $\{x_n\}$ cgs. to $\limsup x_n$. Then \mathbb{R} is complete.

- $X = \mathbb{K}^n$ (Recall 2nd example) complete.

- $X = L^p$ (Recall 3rd example) complete.

- Some examples of noncomplete metric spaces are the half open interval $[0, 1)$ or the set of all rational numbers with the usual metric induced from \mathbb{R} .

If X is a metric space and $E \subset X$, then the completeness of E in the induced metric is related to the closedness of E in X as follows:

- E is complete when E is closed in X .

- If X is complete and E is closed in X , then E is complete.

Theorem (Baire, 1899)

In a metric space X , the intersection of a finite number of dense open sets is dense: if X is complete, then the intersection of a countable number of dense open sets is dense.

Compactness

The concept of compactness is one of the most useful generalization of finiteness.

A subset E of a metric space X is said to be compact (relative to X) if every open cover of X by sets which are open in X has a finite subcover.

Note: If $E \subset Y \subset X$, then E is compact relative to X iff E is compact relative to Y . Thus, we can speak about compact metric spaces in their own right.

Ex: (in a popular language) A city is compact if only a finite no. of watchmen can guard it, even if the watchmen have arbitrarily short sight.

We can define compactness in term of closed sets:

A metric space X is compact iff every collection of closed subsets of X such that any finite subcollection has a nonempty intersection has itself a nonempty intersection.

→ finite intersection property.