

Metric Spaces and Continuous functions

Def. A metric on a set  $X$  is a function  $d: X \times X \rightarrow \mathbb{R}$  such that for  $x, y, z \in X$ , we have

- (i)  $d(x, y) \geq 0$ , and  $d(x, y) = 0 \iff x = y$ .
- (ii)  $d(x, y) = d(y, x)$ , and
- (iii)  $d(x, y) \leq d(x, z) + d(z, y)$ .  $\rightarrow$  triangle inequality.

A metric space is a set  $X$  together with a metric  $d$  on it.

Example.

(a) If ~~is~~  $X$  is any set, and let for any  $x, y \in X$ ,

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Then  $(X, d)$  is a metric space. (discrete metric space).

(b) Let  $X = \mathbb{K}^n$ ,  $n \in \mathbb{N}^+$ . For  $1 \leq p \leq \infty$ ,  $x = (x(1), x(2), \dots, x(n))$ ,  $y = (y(1), y(2), \dots, y(n))$  in  $X$ , let

$$d_p(x, y) = \begin{cases} \left( \sum_{j=1}^n |x(j) - y(j)|^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \max_{1 \leq j \leq n} |x(j) - y(j)| & \text{if } p = \infty. \end{cases}$$

If  $p = 1$  or  $\infty$ , it is well known to us (you can verify that  $d_p$  is a metric on  $\mathbb{K}^n$  and hence  $X = (X, d)$  is a metric space).

If  $1 < p < \infty$ , we can use triangle inequality to show that the property (iii) is satisfied.

Hence  $(X, d): X$  is a metric space,  $1 \leq p \leq \infty$ .

Note: While dealing with these examples, we go through Holder's and Minkowski's inequality.

Note: For  $n=1$ ,  $d \rightarrow$  usual metric.

$\rightarrow$  Now, we generalize the above example (b) into infinite dimensional situation.

(c) For  $1 \leq p < \infty$ , let

$$l^p = \left\{ (x(1), x(2), \dots); \sum_{j=1}^{\infty} |x(j)|^p < \infty \right\}$$

$$l^\infty = \left\{ (x(1), x(2), \dots); \sum_{j=1}^{\infty} |x(j)| < \infty \right\}$$

$l^p, l^\infty$  are linear spaces under componentwise addition and scalar multiplication.

For  $x, y \in l^p$ , let  $d_p(x, y) = \left( \sum_{j=1}^{\infty} |x(j) - y(j)|^p \right)^{1/p}$ , if  $1 \leq p < \infty$

$$d_p(x, y) = \sup_{j=1, 2, \dots} |x(j) - y(j)| \text{ if } p = \infty.$$

H-W  $\rightarrow$  Verify that  $d_p$  is a metric on  $l^p$ .

Hence  $(X, d_p)$  is a metric space.

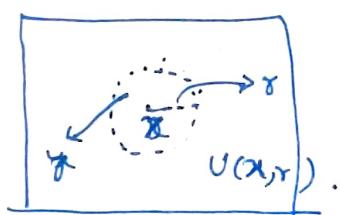
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Open Sets:

Let  $(X, d)$  be a metric space. For any  $x \in X$  and  $r > 0$ , the set

$$U(x, r) = \{y \in X; d(x, y) < r\}.$$

will be called an open ball about  $x$  of radius  $r$ .



The basic notion for a metric space is an open set.

If  $E \subset X$ , we say that  $E$  is open in  $X$  iff for every  $x \in E$ , there exists  $r_x > 0$  (depending on the location of  $x$ ) such that  $U(x, r_x) \subset E$ .

$\rightarrow$  It is easy to see that the sets  $\emptyset$  and  $X$  are open in  $X$ , and that an arbitrary union and finite intersection of open sets in  $X$  is open.

$\Rightarrow E \subset X$  is open in  $X$  iff  $E$  is a union of open balls in  $X$ .

Note:  $\emptyset, X$  are open sets

- For  $E \subset X$ , a point  $x \in E$  is called an ~~open~~ interior point of  $E$  if there exists  $r_x > 0$  such that  $U(x, r_x) \subset E$ .
- Clearly,  $E$  is open iff every point of  $E$  is an interior point of  $E$ .

In general, set of all interior points of  $E$  is called an interior ~~set~~ of  $E$ , denoted by  $E^\circ$  (clearly, largest open set contained in  $E$ ).

- Complementary Concepts:

A set  $E \subset X$  is said to be closed in  $X$  if the complement of  $E$  in  $X$  is open in  $X$ .

Hence  $X, \emptyset$  are closed sets ( $\because \emptyset$  and  $X$  are already open sets). Finite union and arbitrary intersection of closed sets in  $X$  are closed in  $X$ .

For  $E \subset X$ , a point  $x \in X$  is called a limit point (or an accumulation point) of  $E$  if for each  $r > 0$ ,  $U(x, r)$  contains  $y \in E, y \neq x$ . A limit point of  $E$  may or may not belong to  $E$ . In fact, every limit point of  $E$  is in  $E$  iff  $E$  is closed. In general, the set  $\{x \in X; x \in E, \text{ or } x \text{ is a limit point of } E\}$  is called the closure of  $E$ , denoted by  $\bar{E}$ . Clearly it is the smallest closed set containing  $E$ .