

# MA401-Real Analysis and Measure Theory

## Module V-VI

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# Module V

In this Module, we shall study the following:

- Fields
- $\sigma$ -field
- Borel field
- Measure-Inner and outer measure
- Measurable sets and measurable functions
- Measurable space
- Extension of measures
- signed measures
- Jordan-Hahn Decomposition Theorem.

# Open Sets, Closed Sets, and Borel Sets of Real Numbers

**Definition:** A set  $\mathcal{O}$  of real numbers is called **open** provided that for each  $x \in \mathcal{O}$ , there is a  $r > 0$  for which the interval  $(x - r, x + r)$  is contained in  $\mathcal{O}$ .

→ For  $a < b$ , the interval  $(a, b)$  is an open set. Indeed, let  $x \in (a, b)$ . Define  $r = \min\{b - x, x - a\}$ . Observe that  $(x - r, x + r)$  is contained in  $(a, b)$ . Thus,  $(a, b)$  is an open bounded interval and each bounded open interval is of this form.

→ For  $a, b \in \mathbb{R}$ , we define

$$(a, \infty) = \{x \in \mathbb{R} : a < x\}, (-\infty, b) = \{x \in \mathbb{R} : x < b\} \text{ and } (-\infty, \infty) = \mathbb{R}.$$

→ Observe that these sets forms open intervals. We already have seen in previous semester that since each set of real numbers has an infimum and supremum in the set of extended real numbers, then each unbounded open intervals of the above form.

# Open Sets, Closed Sets, and Borel Sets of Real Numbers

**Proposition-1:** The set of real numbers  $\mathbb{R}$  and empty set  $\emptyset$  are open; the intersection of any finite collection of open sets is open; and the union of any collection of open sets is open.

→ We have seen the proof of Proposition-1 in our earlier semester. So no proof... . You need to travel to your old memory.

→ It is not true, however, that the intersection of any collection of open sets is open. We have a beautiful .... burning example. We can see in our eyes.

**Example:** For any natural number  $n$ , let  $\mathcal{O}_n$  be the open interval  $(-1/n, 1/n)$ . Then, by the **Archimedean Property** of  $\mathbb{R}$ ,  $\bigcap_{n=1}^{\infty} \mathcal{O}_n = \{0\}$  and  $\{0\}$  is not an open set.

**Proposition-2:** Every nonempty open set is the disjoint union of a countable collection of open intervals.

# Open Sets, Closed Sets, and Borel Sets of Real Numbers

**Definition:** For a set  $\mathbb{E}$  of real numbers, a real number  $x$  is called a point of closure of  $\mathbb{E}$  provided that every open interval that contains  $x$  also contains a point in  $\mathbb{E}$ . The collection of **point of closure** of  $\mathbb{E}$  is called the **closure** of  $\mathbb{E}$  and denoted by  $\overline{\mathbb{E}}$ .

→ It is clear that we always have  $\mathbb{E} \subseteq \overline{\mathbb{E}}$ . If  $\mathbb{E}$  contains all of its points of closure, that is,  $\mathbb{E} = \overline{\mathbb{E}}$ , the  $\mathbb{E}$  is **closed**.

**Proposition-3:** For a set of real numbers  $\mathbb{E}$ , its closure  $\overline{\mathbb{E}}$  is closed.

Moreover,  $\overline{\mathbb{E}}$  is the smallest closed set that contains  $\mathbb{E}$  in the sense that if  $\mathbb{F}$  is closed and  $\mathbb{E} \subseteq \mathbb{F}$ , then  $\overline{\mathbb{E}} \subseteq \mathbb{F}$ .

**Proposition-4:** A set of real numbers is open if and only if its complements in  $\mathbb{R}$  is closed.

**Remark:** Since  $\mathbb{R} - (\mathbb{R} - \mathbb{E}) = \mathbb{E}$ , it follows from the previous proposition that **a set is closed if and only if its complement is open**. Also, **the empty set  $\phi$  and  $\mathbb{R}$  are closed; the union of any finite collection of closed sets is closed; and the intersection of any collection of closed sets is closed**.

# Open Sets, Closed Sets, and Borel Sets of Real Numbers

**Some Definitions:** A collection of sets  $\{\mathbb{E}_\lambda\}_{\lambda \in \Lambda}$  is said to be a **cover** of a set  $\mathbb{E}$  provided that  $\mathbb{E} \subseteq \bigcup_{\lambda \in \Lambda} \mathbb{E}_\lambda$ . By a subcover of a cover of  $E$  we mean a subcollection of the cover that itself also is a cover of  $\mathbb{E}$ . If each set  $\mathbb{E}_\lambda$  is a cover is open, we call  $\{\mathbb{E}_\lambda\}_{\lambda \in \Lambda}$  an **open cover** of  $\mathbb{E}$ .  $\{\mathbb{E}_\lambda\}_{\lambda \in \Lambda}$  contains only a finite number of sets, we call it a **finite cover**.

We have the following beautiful theorem:

**The Heine-Borel Theorem:** Let  $\mathbb{F}$  be a closed and bounded set of real numbers. Then every open cover of  $\mathbb{F}$  has a finite subcover.

→ Since we are in introduction, I am skipping the proof. You can find the proof in any Real Analysis book.

# Open Sets, Closed Sets, and Borel Sets of Real Numbers

**Definition:  $\sigma$ -algebra** Given a set  $\mathbb{X}$ , a collection of  $\mathbb{A}$  of subsets of  $\mathbb{X}$  is a  $\sigma$ -algebra (of subsets of  $\mathbb{X}$ ) provided (i) the empty set  $\phi$  belongs to  $\mathbb{A}$ ; (ii) the complement in  $\mathbb{X}$  of a set in  $\mathbb{A}$  also belongs to  $\mathbb{A}$ ; (iii) the union of countable collection of sets in  $\mathbb{A}$  also belongs to  $\mathbb{A}$ .

**Example-1:** Given a set  $\mathbb{X}$ , the collection  $\{\phi, \mathbb{X}\}$  is a  $\sigma$ -algebra which has two members and is contained in every  $\sigma$ -algebra of subsets of  $\mathbb{X}$ .

**Example-2:** Another extreme example is the collection of sets  $2^{\mathbb{X}}$  which consists of all subsets of  $\mathbb{X}$  and contains every  $\sigma$ -algebra of subsets of  $\mathbb{X}$ .

**Remark:** For any  $\sigma$ -algebra  $\mathbb{A}$ , we infer from De Morgan's identities that  $\mathbb{A}$  is closed with respect to the formation of intersections of countable collections of sets that belong to  $\mathbb{A}$ ; moreover, since every empty set belongs to  $\mathbb{A}$ ,  $\mathbb{A}$  is closed with respect to the formation of finite unions and finite intersection of sets that belong to  $\mathbb{A}$ .

# Open Sets, Closed Sets, and Borel Sets of Real Numbers

**Proposition-5:** Let  $\mathbb{F}$  be a collection of subsets of a set  $\mathbb{X}$ . Then the intersection  $\mathbb{A}$  of all  $\sigma$ -algebras of subsets of  $\mathbb{X}$  that contains  $\mathbb{F}$  is a  $\sigma$ -algebra that contains  $\mathbb{F}$ . Moreover, it is the smallest  $\sigma$ -algebra of subsets of  $\mathbb{X}$  that contains  $\mathbb{F}$  in the sense that any  $\sigma$ -algebra that contains  $\mathbb{F}$  also contains  $\mathbb{A}$ .

**Definition:** The collection  $\mathbb{B}$  of Borel sets of real numbers is the smallest  $\sigma$ -algebra of sets of real numbers that contains all of the open sets of real numbers.

**Remark:** Every open set is a Borel set and since a  $\sigma$ -algebra is closed with respect to the formation of complements, we have the property that every closed set is a Borel set. Therefore, since each singleton set is closed, every countable set is a Borel set.