

Lebesgue Measure

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Introduction :

- The Riemann integral of a bounded function over a closed, bounded interval is defined using approximations of the function that are associated with partitions of its domain into finite collections of subintervals.
- The generalization of the Riemann integral to the Lebesgue integral will be achieved by using approximations of the function that are associated with decomposition of its domain into finite collections of sets which we call Lebesgue measurable.
- Each interval is Lebesgue measurable.
- The richness of the collection of Lebesgue measurable sets provides better upper and lower approximations of a function, and therefore of its integral, than are possible by just employing intervals.
- This leads to a larger class of functions that are Lebesgue measurable over very general domains and an integral that has better properties.
- We, thus, see : basic concepts of measurable sets and the Lebesgue measure of such a set.
- Let I be an interval. Then
$$l(I) = \begin{cases} \text{difference between the end points : if } I \text{ is bounded} \\ \infty : \text{if } I \text{ is unbounded} \end{cases}$$
- Length is an example of a set function, that is, a function that associates an extended real number to each set in a collection of sets.

- In the case of length, the domain is the collection of all intervals.

$$\text{Length of } [a, b] = l_1 + l_2 + \dots + l_n.$$


$$l = \text{length of } [a, b] = l_1 + l_2 + \dots + l_n.$$

- In this module, we extend the set function length to a large collection of sets of real numbers.

- For instance, the "length" of an "open set" will be the sum of the lengths of the countable number of open intervals of which it is composed.

- What problem we are getting : The collection of sets consisting of intervals and open sets is still too limited for our purposes.

- This forces us to construct a collection of sets called "Lebesgue Measurable sets" and a set function of this collection called "Lebesgue Measure" which is denoted by m .

- The collection of Lebesgue measurable sets is a σ -algebra which contains all open sets and all closed sets.

A collection of subsets of \mathbb{R} is called a σ -algebra provided it contains \mathbb{R} and is closed w.r.t. the formation of complements and countable unions; by De Morgan's Identities such a collection is also closed w.r.t. formation of countable intersections.

- The set function m possesses the following three properties:

1. The measure of an interval is its length:

Every nonempty interval I is Lebesgue measurable and

$$m(I) = l(I).$$

2. Measure is translation invariant:

If E is Lebesgue measurable and y is any number, then the translate of E by y ,

$$E+y = \{x+y; x \in E\},$$

is also Lebesgue measurable and

$$m(E+y) = m(E).$$

3. Measure is countably additive over countable disjoint union of sets:

If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of measurable sets (Lebesgue), then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

This equality holds because we have only disjoint sets. Otherwise, we shall have " \leq " in place of " $=$ ". We shall see this later.

Drawback: It is not possible to construct a set function that possesses the above three properties and is defined for all sets of real numbers. (We shall see this later.)

→ In fact, there is not even a set function defined on all sets of real numbers that possesses the first two properties and is finitely additive. (We shall see this property next week.)

→ What we shall do: We respond to this limitation by constructing a set function on a very rich class of sets that possess the above three properties.

The construction has two stages:

Stage 1. We first construct a set function called Outer-Measure (outer-measure), which we denote by m^* . It is defined for any set, thus, in particular, for any interval.

- The outer measure of any interval is its length.
- Outer measure is translation invariant.
- However, outer measure is not finitely additive. But, it is countable subadditive in the sense that if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, disjoint or not, then

$$m^* \left(\bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Stage 2: The second stage in the construction is to determine what it meant for a set to be Lebesgue measurable and show that the collection of Lebesgue-measurable sets is a σ -algebra containing the open and closed sets. We then construct a set function m^* to the collection of Lebesgue

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measurable sets, denote it by m , and prove
 m is countably additive. We called m Lebesgue
measure.

Lebesgue Outer Measure

Let I be a nonempty interval of real numbers. We define its length, $l(I)$, to be ∞ if I is unbounded, and otherwise define its length to be the difference of its endpoints.

For a set A of real numbers, consider the countable collections $\{I_k\}_{k=1}^{\infty}$ of nonempty, open bounded intervals I_k -cover A , that is, collections for which $A \subseteq \bigcup_{k=1}^{\infty} I_k$.

for each such collection, consider the sum of the lengths of the intervals in the collection. Since the lengths are positive numbers, each sum is uniquely defined independently of the order of terms. On this case, we define the outer measure of A , $m^*(A)$, to be the infimum of all such sums, that is,

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} l(I_k) ; A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

Deductions: 1. $m^*(\emptyset) = 0$.

2. Since any cover of a set B is also a cover of any subset of B , outer measure is monotone, in the sense that

$$\text{if } A \subseteq B, \text{ then } m^*(A) \leq m^*(B).$$

Example: A countable set has outer measure zero. Indeed, let C be a countable set enumerated as $C = \{c_k\}_{k=1}^{\infty}$. Let $\epsilon > 0$. For each $k \in \mathbb{N}$, define $I_k = (c_k - \epsilon/2^{k+1}, c_k + \epsilon/2^{k+1})$. The countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ covers C . Therefore, $0 \leq m^*(C) \leq \sum_{k=1}^{\infty} l(I_k) = \sum_{k=1}^{\infty} \epsilon/2^{k+1} = \epsilon$.

This inequality holds for each $\epsilon > 0$. Hence $m^*(C) = 0$. proved