

Lebesgue Measure

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Introduction :

- The Riemann integral of a bounded function over a closed, bounded interval is defined using approximations of the function that are associated with partitions of its domain into finite collections of subintervals.
- The generalization of the Riemann integral to the Lebesgue integral will be achieved by using approximations of the function that are associated with decomposition of its domain into finite collections of sets which we call Lebesgue measurable.
- Each interval is Lebesgue measurable.
- The richness of the collection of Lebesgue measurable sets provides better upper and lower approximations of a function, and therefore of its integral, than are possible by just employing intervals.
- This leads to a larger class of functions that are Lebesgue measurable over very general domains and an integral that has better properties.
- We, thus, see: basic concepts of measurable sets and the Lebesgue measure of such a set.
- Let I be an interval. Then
$$l(I) = \begin{cases} \text{difference between the endpoints} & : \text{if } I \text{ is bounded} \\ \infty & : \text{if } I \text{ is unbounded} \end{cases}$$
- Length is an example of a set function, that is, a function that associates an extended real number to each set in a collection of sets.

→ In the case of length, the domain is the collection of all intervals.



$$L = \text{length of } [a, b] = l_1 + l_2 + \dots + l_n.$$

→ In this module, we extend the set function length to a large collection of sets of real numbers.

→ For instance, the "length" of an "open set" will be the sum of the lengths of the countable number of open intervals of which it is composed.

→ What problem we are getting: The collection of sets consisting of intervals and open sets is still too limited for our purposes.

→ This forces us to construct a collection of sets called "Lebesgue Measurable sets" and a set function of this collection called "Lebesgue Measure" which is denoted by m .

→ The collection of Lebesgue measurable sets is a σ -algebra which contains all open sets and all closed sets.

A collection of subsets of \mathbb{R} is called a σ -algebra provided it contains \mathbb{R} and is closed w.r.t. to the formation of complements and countable unions; by De Morgan's identities, such a collection is also closed w.r.t. to formation of countable intersections...

• The set function m possesses the following three properties:

1. The measure of an interval is its length:
Every nonempty interval I is Lebesgue measurable and

$$m(I) = l(I).$$

2. Measure is translation invariant:

If E is Lebesgue measurable and y is any number, then the translate of E by y ,

$$E+y = \{x+y; x \in E\},$$

is also Lebesgue measurable and

$$m(E+y) = m(E).$$

3. Measure is countably additive over countable disjoint union of sets:

If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of measurable sets (Lebesgue), then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

their equality holds because we have only disjoint sets. Otherwise; we shall have " \leq " in place of " $=$ ". We shall see this later.

Drawback: It is not possible to construct a set function that possesses the above three properties and is defined for all sets of real numbers. (We shall see this later.)

→ In fact, there is not even a set function defined on all sets of real numbers that possesses the first two properties and is finitely additive. (We shall see this property next week.)

→ What we shall do: We respond to this limitation by constructing a set function on a very rich class of sets that possess the above three properties.

The construction has two stages:

Stage 1. We first construct a set function called Outer-Measure (outer-measure), which we denote by m^* . It is defined for any set, thus, in particular, for any interval.

- The outer measure of any interval is its length.
- Outer measure is translation invariant.
- However, outer measure is not finitely additive. But, it is countable subadditive in the sense that if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, disjoint or not, then

$$m^* \left(\bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Stage 2: The second stage in the construction is to determine what it meant for a set to be Lebesgue measurable and show that the collection of Lebesgue-measurable sets is a σ -algebra containing the open and closed sets. We then construct the set function m^* to the collection of Lebesgue

measurable sets, denote it by m , and prove m is countably additive. We called m Lebesgue measure.

Lebesgue Outer Measure

Let I be a nonempty interval of real numbers. We define its length, $l(I)$, to be ∞ if I is unbounded, and otherwise define its length to be the difference of its endpoints.

For a set A of real numbers, consider the countable collections $\{I_k\}_{k=1}^{\infty}$ of nonempty, open bounded intervals that cover A , that is, collections for which $A \subseteq \bigcup_{k=1}^{\infty} I_k$.

For each such collection, consider the sum of the lengths of the intervals in the collection. Since the lengths are positive numbers, each sum is uniquely defined independently of the order of terms. In this case, we define the outer measure of A , $m^*(A)$, to be the infimum of all such sums, that is,

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} l(I_k) ; A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

Deductions: 1. $m^*(\emptyset) = 0$.

2. Since any cover of a set B is also a cover of any subset of B , outer measure is monotone, in the sense that

$$\text{if } A \subseteq B, \text{ then } m^*(A) \leq m^*(B).$$

Example: A countable set has outer measure zero. Indeed, let C be a countable set enumerated as $C = \{c_k\}_{k=1}^{\infty}$. Let $\epsilon > 0$. For each $k \in \mathbb{N}$, define $I_k = (c_k - \epsilon/2^k, c_k + \epsilon/2^k)$. The countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ covers C . Therefore,

$$0 \leq m^*(C) \leq \sum_{k=1}^{\infty} l(I_k) = \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon.$$

This inequality holds for each $\epsilon > 0$. Hence $m^*(C) = 0$.

Proposition 1 The outer measure of an interval is its length.

Proof. We begin with the case of a closed, bounded interval $[a, b]$.

Let $\epsilon > 0$. Since the open interval $(a-\epsilon, b+\epsilon)$ contains $[a, b]$, we have

$$m^*([a, b]) \leq l((a-\epsilon, b+\epsilon)) = b-a+2\epsilon,$$

which holds for any $\epsilon > 0$. Therefore

$$m^*([a, b]) \leq b-a. \quad \text{--- (1)}$$

It remains to show that

$$m^*([a, b]) \geq b-a \quad \text{--- (2)}$$

so that we shall have $m^*([a, b]) = b-a$.

An equivalent statement showing (2) is that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of open, bounded intervals covering $[a, b]$, then

$$\sum_{k=1}^{\infty} l(I_k) \geq b-a. \quad \text{--- (3)}$$

By the Heine-Borel theorem, any collection of open intervals covering $[a, b]$ has a finite subcollection that also covers $[a, b]$. Choose a natural number n

for which $\{I_k\}_{k=1}^n$ covers $[a, b]$. We will show

(in this case) that $\Rightarrow a \in \bigcup_{k=1}^n I_k$. (note)

$$\sum_{k=1}^n l(I_k) \geq b-a, \quad \text{--- (4)}$$

and therefore (3) holds.

Since $a \in \bigcup_{k=1}^n I_k$, there must be one of the ~~I_k 's~~.

I_k 's that contains a . select such an interval, and denote it by (a_1, b_1) . Thus, we have $a_1 < a < b_1$.

If $b_1 \geq b$, then (4) is established since

$$\sum_{k=1}^n l(I_k) \geq b_1 - a_1 > b - a.$$

Otherwise, $b_1 \in [a, b)$, and since $b_1 \notin (a_1, b_1)$, there is an interval in the collection $\{I_k\}_{k=1}^n$, which we label (a_2, b_2) , distinct from (a_1, b_1) , for which $b_1 \in (a_2, b_2)$; that is, $a_2 < b_1 < b_2$. If $b_2 \geq b$, the inequality (4) is established since

$$\begin{aligned} \sum_{k=1}^n l(I_k) &\geq (b_1 - a_1) + (b_2 - a_2) = b_2 - (a_2 - b_1) - a_1 \\ &> b_2 - a_1 > b - a. \end{aligned}$$

We continue this selection until it terminates, as it must since there are only n intervals in the collection $\{I_k\}_{k=1}^n$. Thus, we obtain a subcollection $\{(a_k, b_k)\}_{k=1}^N$

of $\{I_k\}_{k=1}^n$ for which $a_1 < a$

while

$$a_{k+1} < b_k \text{ for } 1 \leq k \leq N-1,$$

and, since this selection process terminated,

$$b_N > b.$$

Thus,

$$\begin{aligned} \sum_{k=1}^n l(I_k) &\geq \sum_{k=1}^N l((a_i, b_i)) \\ &= (b_N - a_N) + (b_{N-1} - a_{N-1}) + \dots + (b_1 - a_1) \\ &= b_N - (a_N - b_{N-1}) - \dots - (a_2 - b_1) - 1 \\ &> b_N - a_1 > b - a. \end{aligned}$$

Thus, the inequality (4) holds. Consequently, (3) holds, which is equivalent to (2). Hence

$$m^*([a, b]) = b - a.$$

~~Now finally we consider the case that I is any bounded interval.~~

Now, we consider the case that I is any bounded interval. Then given $\epsilon > 0$, there are two closed, bounded intervals J_1 and J_2 such that

$$J_1 \subseteq I \subseteq J_2,$$

while $l(I) - \epsilon < l(J_1)$ and $l(J_2) < l(I) + \epsilon$.

By the equality of outer measure and length for closed, bounded intervals and monotonicity of outer measure,

$$l(I) - \epsilon < l(J_1) = m^*(J_1) \leq m^*(I) \leq m^*(J_2) = l(J_2) < l(I) + \epsilon,$$

which holds for each $\epsilon > 0$. Therefore $l(I) = m^*(I)$.

If I is an unbounded interval, then for each natural number n , there is an interval $J \subseteq I$ with $l(J) = n$. Hence $m^*(I) \geq m^*(J) = l(J) = n$, which holds for each natural number n . Therefore, $m^*(I) = \infty$.

The proposition is proved.

Proposition 2: Outer measure is translation invariant, that is, for any set A and number y ,

$$m^*(A+y) = m^*(A).$$

Proof. Observe that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of open sets, then $\{I_k\}_{k=1}^{\infty}$ covers A if and only if $\{I_k+y\}_{k=1}^{\infty}$ covers $A+y$.

Moreover, if each I_k is an open interval, then each I_k+y is an open interval of the same length and so

$$\sum_{k=1}^{\infty} l(I_k) = \sum_{k=1}^{\infty} l(I_k+y).$$

The conclusion now follows.

Proposition 3. Outer measure is countably sub-additive, that is, if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, disjoint or not, then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Proof. If one of the E_k 's has infinite outer measure, the inequality holds trivially.

We, therefore suppose each of the E_k 's has finite outer measure. Let $\epsilon > 0$. For each natural number k , there is a countable collection $\{I_{k,i}\}_{i=1}^{\infty}$ of open, bounded intervals for which

$$E_k \subseteq \bigcup_{i=1}^{\infty} I_{k,i} \quad \text{and} \quad \sum_{i=1}^{\infty} l(I_{k,i}) < m^*(E_k) + \epsilon/2^k.$$

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Now, $\{I_{k,j}\}$ is a countable collection of open, bounded intervals that covers $\bigcup_{k=1}^{\infty} E_k$; the collection is countable since it is a countable collection of countable collections. Thus, by the definition of outer measure,

$$\begin{aligned}
 m^*\left(\bigcup_{k=1}^{\infty} E_k\right) &\leq \sum_{1 \leq k, i < \infty} l(I_{k,i}) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} l(I_{k,i}) \right) \\
 &< \sum_{k=1}^{\infty} (m^*(E_k) + \epsilon/2^k) \\
 &= \sum_{k=1}^{\infty} m^*(E_k) + \epsilon.
 \end{aligned}$$

~~which~~ ~~holds for each $\epsilon > 0$~~ , ~~it also~~ ~~the~~ which holds for each $\epsilon > 0$. Thus, the proposition is proved.

Remark. If $\{E_k\}_{k=1}^{\infty}$ is any finite collection of sets, disjoint or not, then

$$m^*\left(\bigcup_{k=1}^n E_k\right) \leq \sum_{k=1}^n m^*(E_k).$$

This finite subadditivity property follows from countable subadditivity by taking $E_k = \emptyset$ for $k > n$.