

Sequence and Series of Functions

So far we have considered sequences and series whose terms are numbers. These are the simple cases. We now consider sequence and series whose terms depends on a variable, that is, the terms are real valued functions defined on an interval, which we call a domain. Accordingly, we denote the terms by $\{f_n(x)\}$. So we denote sequences and series of the form $\{f_n(x)\}$ and $\sum f_n(x)$.

Pointwise Convergence

(For sequences) \rightarrow Suppose $\{f_n(x)\}$, $n \geq 1, 2, \dots$ is a sequence of functions, defined on an interval $I := a \leq x \leq b$. To each point $\xi \in I$, there corresponds a sequence of numbers $\{f_n(\xi)\}$ with term $f_1(\xi), f_2(\xi), f_3(\xi), \dots, f_n(\xi), \dots$.

Further, let us suppose that the sequence of numbers $\{f_n(\xi)\}$ converges for every $\xi \in I$.

So, let $\{f_n(\xi)\}$ converges to $f(\xi)$.

In this way, let all the sequence of all points x_1, x_2, \dots of I converges to $f(x_1), f(x_2), \dots, f(x_n), \dots \dots \dots$ (1)

\rightarrow We now define, in a natural way, a real valued function f , with the domain I and the range set defined by (1), so that

$f(\eta)$ for $\eta \in I$ is $\lim f_n(\eta)$.

\rightarrow Thus,

$$f(x) = \lim_{n \rightarrow \infty} f_n(x), \quad \text{if } x \in I := a \leq x \leq b.$$

→ The function f , so defined, is referred to as the limit or the point-wise limit of the sequence $\{f_n\}$ on $I := [a, b]$, the sequence $\{f_n(x)\}$ is said to be point-wise convergent to f on $[a, b]$.

(for review) Similarly, if the series $\sum f_n$ converges for every point $x \in I$, and we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \forall x \in I := [a, b],$$

then the function f is called the sum or the point-wise sum of the series $\sum f_n(x)$ on $[a, b]$.

→ Thus, if f is the pointwise limit of a sequence of functions $\{f_n\}$ defined on $[a, b]$, then to each $\epsilon > 0$ and to each $x \in [a, b]$, there corresponds an integer m such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq m.$$

Remarks. What about boundedness, continuity, differentiability, integrability of the functions $f_n(x), n=1, 2, \dots$? Now, not in the syllabus. We shall see them later.

Few Examples.

1. The geometric series ~~converges to~~
 ~~$1 + x + x^2 + x^3 + \dots$~~ $1 + x + x^2 + x^3 + \dots, \quad -1 < x < 1$

convert to $(1-x)^{-1}$.

2. Consider the series

$$\sum_{n=0}^{\infty} f_n(x), \text{ where } f_n(x) = \frac{x^2}{(1+x^2)^n}, \quad x \text{ real.}$$

At $x=0$, each $f_n(x)=0$, so that the sum of the series is $f(0)=0$.

For $x \neq 0$, it is in the form of geometric series with common ratio $\frac{1}{1+x^2}$, so that its sum function $f(x)$ is

$$f(x) = 1 + x^2.$$

Hence

$$f(x) = \begin{cases} 1 + x^2, & x \neq 0 \\ 0, & x=0 \end{cases}$$

Each term of the series is continuous but the sum f is not.
(Verify that f is not continuous at $x=0$).

3. The sequence $\{f_n\}$, where

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}, x \text{ real}$$

has the limit

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{\sin nx}{\sqrt{n}} = 0.$$

So $f'(x)=0$ as hence $\underline{f'(0)=0}$.

$$\text{But, } f'_n(x) = \sqrt{n} \cos nx$$

so that $\underline{f'_n(0)} = \sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$.

Thus, at $x=0$, the sequence $\{f'_n(x)\}$ diverges whereas the limit function $f'(x)=0$, i.e., the limit of the differentials is not equal to the differentials of the limit.

4. Consider the sequence $\{f_n\}$, where

$$f_n(x) = n^2(1-x^2)^n, 0 \leq x \leq 1, n=1, 2, 3, \dots$$

For $0 < x \leq 1$, $\lim_{n \rightarrow \infty} f_n(x) = 0$.

At $x=0$, each $f_n(0)=0$, so that $\lim_{n \rightarrow \infty} f_n(0)=0$.

Then, the limit function

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0 \text{ for } 0 \leq x \leq 1.$$

Then $\int_0^1 f(x) dx = 0$.

On the other hand,

$$\int_0^1 f_n(x) dx = \int_0^1 nx(1-x^2)^n dx = \frac{n}{2n+2}$$

so that

$$\lim_{n \rightarrow \infty} \left\{ \int_0^1 f_n(x) dx \right\} = \frac{1}{2}.$$

(not inserting the
limit inside the
integration)

$$\text{Then, } \lim_{n \rightarrow \infty} \left\{ \int_0^1 f_n(x) dx \right\} \neq \int_0^1 f(x) dx = \int_0^1 \left\{ \lim_{n \rightarrow \infty} f_n(x) \right\} dx.$$

Then, the limit of the integrals is not equal to the
integral of the limit.

In other words, a sequence of integrals may not
converge to the integral of the limit of the sequence.

A big question: Under what supplementary conditions,
~~does~~ or does f_n or other properties of f_n are
transferred to the limit function f .

A concept of great importance in this respect
is that known as Uniform Convergence of a sequence
(series) in its domain of definition [a, b].

Uniform Convergence on an interval

A sequence of functions $\{f_n\}$ is said to converge uniformly on an interval $[a, b]$ to a function f if for any $\epsilon > 0$ and for all $x \in [a, b]$ there exists an integer N (independent of x but dependent on ϵ) such that for all $x \in [a, b]$,

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N. \quad \dots \quad (*)$$

→ Every uniformly convergent sequence is pointwise convergent and the uniform limit function is same as the pointwise limit function.

Difference between two concept (Pointwise & Uniform convergence):

<u>Point-wise</u>	<u>Uniform</u>
for each $\epsilon > 0$ and each $x \in \mathbb{R}$ there exists an integer $N(\epsilon, x)$ such that $(*)$ holds for $n \geq N$.	for each $\epsilon > 0$, it is possible to find an integer $N(\epsilon)$ such that $(*)$ holds for $n \geq N$.

Remark-1 Uniform convergence \Rightarrow Pointwise convergence,
not vice-versa.

A series of functions $\sum f_n$ is said to converge uniformly on $[a, b]$ if the sequence $\{S_n\}$ of its partial sums,

defined by $S_n = \sum_{i=1}^n f_i(x)$

converges uniformly on $[a, b]$.

→ Thus, a series of functions $\sum f_n$ converges uniformly to f on $[a, b]$ if for $\epsilon > 0$ and all $x \in [a, b]$, there exists an integer N (independent of x and dependent on ϵ) such that for all $x \in [a, b]$,

$$|f_1(x) + f_2(x) + \dots + f_n(x)| < \epsilon \quad \text{for } n \geq N.$$

Cauchy's criterion for uniform convergence:

Theorem 1. A sequence of functions $\{f_n\}$ defined on $[a, b]$ converges uniformly on $[a, b]$ if and only if for every $\epsilon > 0$ and for all $x \in [a, b]$, there exists an integer N such that

that

$$|f_{n+p}(x) - f_n(x)| < \epsilon \quad \forall n \geq N, p \geq 1.$$

Theorem 2. A series of functions $\sum f_n$ defined on $[a, b]$ converges uniformly if and only if for every $\epsilon > 0$ and for all $x \in [a, b]$, there exists an integer N such that

$$|f_{m+1}(x) + f_{m+2}(x) + \dots + f_n(x)| < \epsilon \quad \forall n, m \geq N.$$

Example 5 $\{f_n(x)\}$, $x \in \mathbb{R}$, where $f_n(x) = \frac{nx}{1+n^2x^2}$.

We shall test the uniform convergence.

The sequence converges pointwise to f , where

$$f(x) = 0 \quad \forall \text{ real } x.$$

Let $\{f_n\}$ converges uniformly in any interval $[a, b]$, so that the pointwise limit is also the uniform limit. Therefore, for ~~any~~ given $\epsilon > 0$, there exists an integer N such that for all $x \in [a, b]$,

$$\left| \frac{nx}{1+n^2x^2} - 0 \right| < \epsilon \quad \forall n \geq N.$$

If we take $\epsilon = \frac{1}{3}$, and m is an integer greater than N such that $y_m \in [a, b]$, we find on taking $n = m$ and $x = y_m$, that

$$\frac{nx}{1+n^2x^2} = \frac{1}{2} + \frac{1}{3} = \epsilon.$$

Thus, we arrive at a contradiction and so the sequence is not uniformly convergent in the interval $[a, b]$, which contains the point m .

But, since $y_m > 0$, the series contains 0. Hence the sequence is not uniformly convergent on any interval containing 0.

A test for uniform convergence of Sequences:

Theorem 3. Let $\{f_n\}$ be a sequence of functions, such that

$$\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad x \in [a, b]$$

and let

$$M_n = \sup_{x \in [a, b]} |f_n(x) - f(x)|.$$

Then $f_n \rightarrow f$ uniformly on $[a, b]$ if $M_n \rightarrow 0$ as $n \rightarrow \infty$.

Example - 6. (We consider Ex. 5 with some modifications as per Theorem 3)

Show that the sequence $\{f_n\}$, where

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

is not uniformly convergent on any interval containing 0.

Proof. Here $\lim_{n \rightarrow \infty} f_n(x) = 0 \quad \forall x$.

Verify that $\frac{nx}{1+n^2x^2}$ attains the maximum y_2 at $x = \frac{1}{n}$;

$\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$. Let us take an interval containing 0.

$$\text{Then, } M_n = \sup_{x \in [a, b]} |f_n(x) - f(x)|$$

$$= \sup_{x \in [a, b]} \left| \frac{nx}{1+n^2x^2} \right| = y_2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence the sequence $\{f_n\}$ is not uniformly convergent (we have applied Theorem 3). (proved).

Test of Uniform Convergence of Series

Weierstrass's M-Test

A series of functions $\sum f_n$ converge uniformly (and absolutely) on $[a, b]$ if there exists a convergent series $\sum M_n$ of positive numbers such that for all $x \in [a, b]$

$$|f_n(x)| \leq M_n \text{ for all } n.$$

normally
converges

if applied to
all series
in everyday life.

Remark

→ Non convergence of $\sum M_n$ need not imply anything on $\sum f_n$.

Illustrations on Weierstrass's M-Test :

i) The series $\sum r^n \cos n\theta, \sum r^n \sin n\theta, \sum r^n \cos n^2\theta, \sum r^n \sin(n^2\theta)$, $0 < r < 1$, converges uniformly for all real values of θ .
 Hint: Take $M_n = r^n$. Note: we already have the condition $0 < r < 1$.

ii) The series $\sum \frac{a_n x^n}{1+x^{2n}}, \sum \frac{a_n x^{2n}}{1+x^{2n}}$, converge uniformly for all real values of x , if $\sum a_n$ is absolutely convergent

iii) $\sum \frac{\sin(x^2 + n^2 x)}{n(n+1)}$ is uniformly convergent for all real x .
 Hint: set $M_n = \frac{1}{n(n+1)}$.

iv) $\sum \frac{\cos n\theta}{n^p}$ is uniformly and absolutely convergent for all real values of θ and $p > 1$.

Hint : $M_n = Y_{np}$.

v) $\sum \frac{1}{n^{\delta} x}$ is uniformly convergent in $[1+\delta, \infty), \delta > 0$.