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Positive Periodic Solutions for Systems of Nonlinear Differential Equations with Discrete and Distributed Delays

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Abstract

We consider the following system of equations

$$x_i'(t) = x_i(t)[r_i(t) - f_i(t, x(t), y(t))],\ i = 1, 2, \cdots, n,$$
$$y_j'(t) = y_j(t)[-\hat{r}_j(t) + g_j(t, x(t), y(t))],\ j = 1, 2, \cdots, m,$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t)), y(t) = (y_1(t), y_2(t), \cdots, y_m(t))$, $f_i, g_j \in C([0, T] \times \mathbb{R}_+^{n+m}, \mathbb{R}_+)$ for $i = 1, 2, \cdots, n,\quad j = 1, 2, \cdots, m$ are T-periodic functions in x and y, $T > 0$ is a real number and $\mathbb{R}_+ = [0, \infty)$.

Abstract

Based on a version of fixed point theory in a cone due to Gustafson and Schmitt, we obtain sufficient conditions for the existence of positive periodic solutions of the above system. As an application, we prove that only the positive periodic coefficients are required in order to obtain a positive periodic solution of a certain Lotka-Volterra type ecological model with discrete and distributed delays.

Introduction

- System of functional differential equations with periodic delays appear in many models of biological and ecological systems.
- One such model is system of competition type model. The reader may refer the following monographs for such models.

1 . K. Gopalsamy; *Stability and Oscillation in Delay Differential Equations of Population Dynamics*, Kluwer Academic Press, Boston, 1992.

2 . M. Kot; Elements of Mathematical Ecology, Cambridge University Press, 2001.

3 . Y. Kuang; *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, New York, 1993.

4 . J. D. Murray; Mathematical Biology I: An Introduction, Springer International Edition, 2002

Introduction

- One of the important model in ecological dynamics is the n-species Gilpin-Ayala type competitive ecological population model with discrete and distributed delays

$$\begin{aligned} x_i'(t) = x_i(t) \Bigg[r_i(t) - \sum_{k=1}^{n} a_{ik}(t) x_k^{\theta_k}(t - \tau_{ik}(t)) - \\ \sum_{k=1}^{n} c_{ik}(t) \int_{-\infty}^{0} K_{ik}(s) x_k^{\theta_k}(t+s)\, ds \Bigg], \end{aligned} \tag{1.1}$$

where $i = 1, 2, \cdots, n$, $r_i, a_{ik}, c_{ik} \in C(\mathbb{R}, (0, \infty))$, and $\tau_{ik} \in C(\mathbb{R}, \mathbb{R})\,(i, k = 1, 2, \cdots, n)$ are T-periodic functions, $\theta_k > 0$ for $k = 1, 2, \cdots, n$ are constants, and $K_{ik} \in C((-\infty, 0], (0, \infty))$ with $\int_{-\infty}^{0} K_{ik}(s)\, ds = 1$ and $T > 0$ is a real number.

Introduction (Cont...)

System (1.1) is a particular case of the following system of equations with discrete and distributed delays

$$\begin{cases} x_i'(t) = x_i(t)[r_i(t) - F_i(t, x(t), y(t))], \ i = 1, 2, \cdots, n, \\ y_j'(t) = y_j(t)[-\hat{r}_j(t) + \hat{F}_j(t, x(t), y(t))], \ j = 1, 2, \cdots, m, \end{cases} \tag{1.2}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t)), y(t) = (y_1(t), y_2(t), \cdots, y_m(t))$.

$$
\begin{aligned}
F_i(t,x(t),y(t)) &= \sum_{k=1}^{n} a_{ik}(t)x_k^{\theta_k}(t-\tau_{ik}(t)) + \sum_{l=1}^{m} b_{il}(t)y_l^{\rho_l}(t-\sigma_{il}(t)) \\
&\quad + \sum_{k=1}^{n} c_{ik}(t)\int_{-\infty}^{0} K_{ik}(s)x_k^{\theta_k}(t+s)\,ds \\
&\quad + \sum_{l=1}^{m} d_{il}(t)\int_{-\infty}^{0} L_{il}(s)y_l^{\rho_l}(t+s)\,ds, \\
\hat{F}_j(t,x(t),y(t)) &= \sum_{k=1}^{n} \hat{a}_{kj}(t)x_k^{\theta_k}(t-\hat{\tau}_{kj}(t)) + \sum_{l=1}^{m} \hat{b}_{lj}(t)y_l^{\rho_l}(t-\hat{\sigma}_{lj}(t)) \\
&\quad + \sum_{k=1}^{n} \hat{c}_{kj}(t)\int_{-\infty}^{0} \hat{K}_{kj}(s)x_k^{\theta_k}(t+s)\,ds \\
&\quad + \sum_{l=1}^{m} \hat{d}_{lj}(t)\int_{-\infty}^{0} \hat{L}_{lj}(s)y_l^{\rho_l}(t+s)\,ds,
\end{aligned}
$$

Here,

- $r_i, \hat{r}_j, a_{ik}, \hat{a}_{kj}, b_{il}, \hat{b}_{lj}, c_{ik}, \hat{c}_{kj}, d_{il}, \hat{d}_{lj} \in C(\mathbb{R}, (0, \infty))$.
- $\tau_{ik}, \sigma_{il}, \hat{\tau}_{kj}, \hat{\sigma}_{lj} \in C(\mathbb{R}, \mathbb{R})$, $i, k = 1, 2, \cdots, n$; $j, l = 1, 2, \cdots, m$ are T-periodic functions.
- $\theta_i > 0$, $i = 1, 2, \cdots, n$, and $\rho_j > 0$, $j = 1, 2, \cdots, m$ are constants.
- $K_{ik}, L_{il}, \hat{K}_{kj}, \hat{L}_{lj} \in C((-\infty, 0], (0, \infty))$ with

$$\int_{-\infty}^{0} K_{ik}(s)\, ds = \int_{-\infty}^{0} \hat{K}_{kj}(s)\, ds = \int_{-\infty}^{0} L_{il}(s)\, ds = \int_{-\infty}^{0} \hat{L}_{lj}(s)\, ds = 1.$$

Set

$$\delta_i = e^{-\theta_i \int_0^T r_i(s)\, ds}, \quad \hat{\delta}_j = e^{\rho_j \int_0^T \hat{r}_j(s)\, ds},$$

$$\Gamma_i = \frac{\theta_i}{1-\delta_i} \int_0^T \left[\sum_{k=1}^{n} (a_{ik}(s) + c_{ik}(s)) + \sum_{l=1}^{m} (b_{il}(s) + d_{il}(s)) \right] ds,$$

$$\hat{\Gamma}_j = \frac{\rho_j \hat{\delta}_j}{\hat{\delta}_j - 1} \int_0^T \left[\sum_{k=1}^{n} (\hat{a}_{kj}(s) + \hat{c}_{kj}(s)) + \sum_{l=1}^{m} (\hat{b}_{lj}(s) + \hat{d}_{lj}(s)) \right] ds,$$

$$\Pi_i = \frac{\theta_i \delta_i^2}{1-\delta_i} \int_0^T \left[\sum_{k=1}^{n} \delta_k (a_{ik}(s) + c_{ik}(s)) + \sum_{l=1}^{m} \frac{1}{\hat{\delta}_l} (b_{il}(s) + d_{il}(s)) \right] ds,$$

and

$$\hat{\Pi}_j = \frac{\rho_j}{\hat{\delta}_j(\hat{\delta}_j - 1)} \int_0^T \left[\sum_{k=1}^{n} \delta_k (\hat{a}_{kj}(s) + \hat{c}_{kj}(s)) + \sum_{l=1}^{m} \frac{1}{\hat{\delta}_l} (\hat{b}_{lj}(s) + \hat{d}_{lj}(s)) \right] ds;$$

then Zhao and Ren [18] proved that if $\Gamma < 1$, then (1.2) has a positive T-periodic solution, where

$$\Gamma = \min \left\{ \frac{1}{\Gamma_1}, \frac{1}{\Gamma_2}, \cdots, \frac{1}{\Gamma_n}, \frac{1}{\hat{\Gamma}_1}, \frac{1}{\hat{\Gamma}_2}, \cdots, \frac{1}{\hat{\Gamma}_m} \right\}. \tag{1.3}$$

Theorem 1.1 ([4, 6])

Let X be a Banach space, $P \subset X$ a cone, $0 < r < R$,

$$D = \{x \in P : r \leq \|x\| \leq R\},$$

and $\Phi : D \to P$ be a compact continuous operator such that

(a) $x \in D, \mu > 1, \; x = \mu \Phi x \Longrightarrow \|x\| \neq R.$

(b) $x \in D, \mu \in (0, 1), \; x = \mu \Phi x \Longrightarrow \|x\| \neq r.$

(c) $\inf_{\|x\|=R} \|\Phi x\| \neq 0.$

Then Φ has a fixed point in D.

Introduction (Cont... Appl. Fixed Poin Thm....)

- Applications of Theorem 1.1 and its other versions to obtain sufficient conditons for the existence of positive solutions of second order boundary value problems can be found in:

1. V. Anuradha, D. D. Hai and R. Shivaji; Existence results for superlinear semipositone BVP's, Proc. Amer. Math. Soc. 124(3)(1996), 757-763.
2. J. Gatica and Y. Kim; Positive solutions of superlinear and sublinear boundary value problems, Korean J. Math. 25(1)(2017), 37-43.
3. X. Yang; Green's function and positive solutions for higher order ODE, Appl. Math. Comput. 136(2003), 379-393.

- Application of Theorem 1.1 on the first order differential equation is relatively scarce in the literature.
- In a recent work, D. D. Hai and Chuanxi Qian (On positive periodic solutions for nonlinear dalayed differential equations, Mediterr. J. Math. 13(2016), 1641-1651.) used Theorem 1.1 to obtain several sufficient conditions on the existence of positive T-periodic solutions of the following delayed differential equation

$$x'(t) = a(t)g(x(t))x(t) - \lambda b(t)f(x(t - \tau(t))),$$

where λ is a positive parameter, $a, b, \tau \in C(\mathbb{R}, \mathbb{R})$ are T-periodic functions with $a, b \geq 0, z, b \not\equiv 0, f, g \in C([0, \infty), [0, \infty))$.

Introduction (Cont...)

We shall apply Theorem 1.1 to the following system of equations

$$\begin{cases} x_i'(t) = x_i(t)[r_i(t) - f_i(t, u(t), v(t))], \ i = 1, 2, \cdots, n, \\ y_j'(t) = y_j(t)[-\hat{r}_j(t) + g_j(t, u(t), v(t))], \ j = 1, 2, \cdots, m, \end{cases} \tag{1.4}$$

where $u(t) = (x_1(t), x_2(t), \cdots, x_n(t)), v(t) = (y_1(t), y_2(t), \cdots, y_m(t))$, $f_i, g_j \in C([0, T] \times \mathbb{R}_+^{n+m}, \mathbb{R}_+)$ for $i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m$ are T-periodic functions in u and v, $T > 0$ is a real number and $\mathbb{R}_+ = [0, \infty)$.

Main Result

Theorem 1.2

Assume that there exist constants $0 < r < R$ *such that*

(H1) $\int_0^T f_i(s, u(s), v(s))\, ds \leq (1 - \sigma_i)\|x\|$ *for* $0 \leq x \leq r$, $i = 1, 2, \cdots, n$,

(H2) $\int_0^T g_j(s, u(s), v(s))\, ds \leq \frac{\hat{\sigma}_j - 1}{\hat{\sigma}_j}\|x\|$ *for* $0 \leq x \leq r$, $j = 1, 2, \cdots, m$,

(H3) $\int_0^T f_i(s, u(s), v(s))\, ds \geq \frac{(1-\sigma_i)}{\sigma_i}\|x\|$ *for* $\|x\| = R$, $i = 1, 2, \cdots, n$ *and*

(H4) $\int_0^T g_j(s, u(s), v(s))\, ds \geq (\hat{\sigma}_j - 1)\|x\|$ *for* $\|x\| = R$, $j = 1, 2, \cdots, m$.

Then the system (1.4) has a positive T*-periodic solution.*

Main Result

A direct application application of Theorem 1.1 or an application of our Theorem 1.2 yields the following theorem.

Theorem 1.3

The system (1.2) has a positive T-periodic solution.

Theorem 1.3 shows that the condition $\Gamma < 1$, considered by K. Zhao and Y. Ren (Existence of positive periodic solutions for a class of Gilpin-Ayala ecological models with discrete and distributed tide delays, Adv. Difference Equations, 331; 2017(2017), DOI 10.1186/s13662-017-1386-9) is not necessary. As a consequence of Theorem 1.3, we have the following corollary.

Corollary 1.4

The model (1.1) has a positive T-periodic solution.

Clearly,
$x(t) = (u(t), v(t))^T = (x_1(t), x_2(t), \cdots, x_n(t), y_1(t), y_2(t), \cdots, y_m(t))^T$ is a T-periodic solution of the system (1.4) if and only if it satisfies the the integral system

$$\begin{cases} x_i(t) = \int_t^{t+T} H_i(t,s) f_i(s, u(s), v(s))\, ds, \ i = 1, 2, \cdots, n \\ y_j(t) = \int_t^{t+T} \hat{H}_j(t,s) g_j(s, u(s), v(s))\, ds, \ j = 1, 2, \cdots, m, \end{cases} \tag{2.1}$$

where $H_i(t,s), i = 1, 2, \cdots, n$ and $\hat{H}_j(t,s), j = 1, 2, \cdots, m$ are the Green's functions, given by

$$H_i(t,s) = \frac{e^{-\int_t^s r_i(\tau)\, d\tau}}{1 - e^{\int_0^T r_i(\tau)\, d\tau}}, \ s \in [t, t+T], \ i = 1, 2, \cdots, n, \tag{2.2}$$

$$\hat{H}_j(t,s) = \frac{e^{\int_t^s \hat{r}_j(\tau)\, d\tau}}{e^{\int_0^T \hat{r}_j(\tau)\, d\tau} - 1}, \ s \in [t, t+T], \ j = 1, 2, \cdots, m. \tag{2.3}$$

Proof of Theorem 1.2

By the periodicity of r_i and $\hat{r}_j$, we have

$$H_i(t+T, s+T) = H_i(t,s), \hat{H}_j(t+T, s+T) = \hat{H}_j(t,s).$$

Set $\sigma_i = e^{-\int_0^T r_i(\tau)\, d\tau}$, $\hat{\sigma}_j = e^{\int_0^T \hat{r}_j(\tau)\, d\tau}$; then the Green's functions $H_i(t,s)$ and $\hat{H}_j(t,s)$ satisfies the inequalities

$$\frac{\sigma_i}{1-\sigma_i} \leq H_i(t,s) \leq \frac{1}{1-\sigma_i}, \;\; i = 1, 2, \cdots, n, \tag{2.4}$$

and

$$\frac{1}{\hat{\sigma}_j - 1} \leq \hat{H}_j(t,s) \leq \frac{\hat{\sigma}_j}{\hat{\sigma}_j - 1}, \;\; j = 1, 2, \cdots, m. \tag{2.5}$$

Proof of Theorem 1.2

Let

$$X = \{x \in C(\mathbb{R}^{n+m}, \mathbb{R}) : x(t+T) = x(t)\},$$

endowed with the norm

$$\|x\| = \max_{1 \le i \le n+m} |x_i|_0,$$

where

$$|x_i|_0 = \sup_{t \in [0,T]} \{|x_i(t)|\}, i = 1, 2, \cdots, n+m.$$

Then X is a Banach space. In view of (2.4) and (2.5), we define a cone N on X as

$$N = \{x = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m) \in X : x_i(t) \ge \sigma_i |x_i|_0, y_j(t) \ge \frac{1}{\hat{\sigma}_j} |y_j|_0,$$
$$t \in [0, T]\}$$

Proof of Theorem 1.2

Define an operator Θ on X by

$$(\Theta x)(t) = ((\Theta_1 x)(t), (\Theta_2 x)(t), \cdots, (\Theta_n x)(t), (\Delta_1 x)(t), (\Delta_2 x)(t), \cdots, (\Delta_m x)(t)) \tag{2.6}$$

where

$$(\Theta_i x)(t) = \int_t^{t+T} H_i(t,s) f_i(s, u(s), v(s))\, ds, \ i = 1, 2, \cdots, n \tag{2.7}$$

and

$$(\Delta_j x)(t) = \int_t^{t+T} \hat{H}_j(t,s) g_j(s, u(s), v(s))\, ds, \ j = 1, 2, \cdots, m. \tag{2.8}$$

Now, for $0 < r < R$, we consider the set

$$D = \{x \in N : r \leq \|x\| \leq R\}.$$

Using (2.4) and (2.5), we can show that $\Theta : D \to N$, compact and continuous. We shall use Theorem 1.1 to prove our theorem.

Proof of Theorem 1.2

First, suppose that $x \in D$ with $x = \mu\Theta x$, and $\mu \in (0, 1)$. We claim that $\|x\| \neq r$. If this is not true, then $\|x\| = r$. For any $t \in [0, T]$, we have

$$|(\Theta_i x)(t)| = \int_t^{t+T} H_i(t,s) f_i(s, u(s), v(s))\, ds, \leq \frac{1}{1-\sigma_i} \int_0^T f_i(s, u(s), v(s))\, ds, \\ < \|x\| \qquad (2.9)$$

and

$$|(\Delta_j x)(t)| = \int_t^{t+T} \hat{H}_j(t,s) g_j(s, u(s), v(s))\, ds, \leq \frac{\hat{\sigma}_j}{\hat{\sigma}_j - 1} \int_0^T g_j(s, u(s), v(s))\, ds, \\ < \|x\|. \qquad (2.10)$$

From the inequalities (2.9) and (2.10), we obtain $r = \|x\| = \mu\|\Theta x\| < \|x\| = r$, a contradiction. Hence $\|x\| \neq r$. Thus, the condition $(\mathbf{b})$ of Theorem 1.1 is satisfied.

Next, suppose that $x \in D$ and $x = \mu\Theta x$ with $\mu > 1$. We claim that $\|x\| \neq R$. If possible, suppose that $\|x\| = R$. Then, for any $t \in [0, T]$, we have

$$|(\Theta_i x)(t)| = \int_t^{t+T} H_i(t,s) f_i(s, u(s), v(s))\, ds, \geq \frac{\sigma_i}{1-\sigma_i} \int_0^T f_i(s, u(s), v(s))\, ds, \\ > \|x\| \tag{2.11}$$

and

$$|(\Delta_j x)(t)| = \int_t^{t+T} \hat{H}_j(t,s) g_j(s, u(s), v(s))\, ds, \geq \frac{1}{\hat{\sigma}_j - 1} \int_0^T g_j(s, u(s), v(s))\, ds, \\ > \|x\|. \tag{2.12}$$

Proof of Theorem 1.2: Completed.

Hence, from the inequalities (2.11) and (2.12), we have
$R = \|x\| = \mu\|\Phi x\| > \|x\| = R$, a contradiction. Hence $\|x\| \neq R$.

Thus, the condition (**a**) of Theorem 1.1 is satisfied.

Furthermore, the conditions (2.11) and (2.12) implies that
$\inf_{\|x\|=R} \|\Phi x\| > R \neq 0$ holds.

This proves the condition (**c**) of Theorem 1.1.

By Theorem 1.1, the system (1.4) has a positive T-periodic solution. □

Proof of Theorem 1.3

Set $x_i^{\theta_i}(t) = u_i(t)$ and $y_j^{\rho_j}(t) = v_j(t)$; then

$$x(t) = (u(t), v(t))^T = (u_1(t), u_2(t), \cdots, u_n(t), v_1(t), v_2(t), \cdots, v_m(t))^T \tag{3.1}$$

is a positive T-periodic solution of the system

$$\begin{cases} u_i'(t) = \theta_i u_i(t)[r_i(t) - F_i(t, u(t), v(t))], \ i = 1, 2, \cdots, n \\ v_j'(t) = \rho_j v_j(t)[-\hat{r}_j(t) + \hat{F}_j(t, u(t), v(t))], \ j = 1, 2, \cdots, m, \end{cases} \tag{3.2}$$

if and only if

$$(u(t), v(t)) = (u_1^{\frac{1}{\theta_1}}(t), u_2^{\frac{1}{\theta_2}}(t), \cdots, u_n^{\frac{1}{\theta_n}}(t), v_1^{\frac{1}{\rho_1}}(t), v_2^{\frac{1}{\rho_2}}(t), \cdots, v_m^{\frac{1}{\rho_m}}(t))^T$$

is a positive T-periodic solution of the system (1.2).

Proof of Theorem 1.3

$$
\begin{aligned}
F_i(t, u(t), u(t)) = &\sum_{k=1}^{n} a_{ik}(t) u_k(t - \tau_{ik}(t)) + \sum_{l=1}^{m} b_{il}(t) v_l(t - \sigma_{il}(t)) \\
&+ \sum_{k=1}^{n} c_{ik}(t) \int_{-\infty}^{0} K_{ik}(s) u_k(t+s)\, ds \\
&+ \sum_{l=1}^{m} d_{il}(t) \int_{-\infty}^{0} L_{il}(s) v_l(t+s)\, ds, \\
\hat{F}_j(t, u(t), v(t)) = &\sum_{k=1}^{n} \hat{a}_{kj}(t) u_k(t - \hat{\tau}_{kj}(t)) + \sum_{l=1}^{m} \hat{b}_{lj}(t) v_l(t - \hat{\sigma}_{lj}(t)) \\
&+ \sum_{k=1}^{n} \hat{c}_{kj}(t) \int_{-\infty}^{0} \hat{K}_{kj}(s) u_k(t+s)\, ds \\
&+ \sum_{l=1}^{m} \hat{d}_{lj}(t) \int_{-\infty}^{0} \hat{L}_{lj}(s) v_l(t+s)\, ds.
\end{aligned}
$$

Proof of Theorem 1.3

Further, assuming that $x(t)$, given in (3.1), is periodic with period T, we see that a positive periodic solution $x(t)$ of (3.2) is equivalent to a positive solution $x(t)$ of the integral system

$$\begin{cases} u_i(t) = \theta_i \int_t^{t+T} G_i(t,s)u_i(s)F_i(s,u(s),v(s))\,ds, \ i = 1,2,\cdots,n \\ v_j(t) = \rho_j \int_t^{t+T} \hat{G}_j(t,s)v_j(s)\hat{F}_j(s,u(s),v(s))\,ds, \ j = 1,2,\cdots,m, \end{cases} \tag{3.3}$$

where $G_i(t,s), i = 1,2,\cdots,n$ and $\hat{G}_j(t,s), j = 1,2,\cdots,m$ are the Green's functions, given by

$$G_i(t,s) = \frac{e^{-\theta_i \int_t^s r_i(z)\,dz}}{1 - e^{-\theta_i \int_0^T r_i(z)\,dz}}, \ s \in [t, t+T], \ i = 1,2,\cdots,n, \tag{3.4}$$

$$\hat{G}_j(t,s) = \frac{e^{\rho_j \int_t^s \hat{r}_j(z)\,dz}}{e^{\rho_j \int_0^T \hat{r}_j(z)\,dz} - 1}, \ s \in [t, t+T], \ j = 1,2,\cdots,m. \tag{3.5}$$

Proof of Theorem 1.3

In this case, $G_i(t+T, s+T) = G_i(t,s)$, $\hat{G}_j(t+T, s+T) = \hat{G}_j(t,s)$, and the Green's functions $G_i(t,s)$ and $\hat{G}_j(t,s)$ satisfies the inequalities

$$\frac{\delta_i}{1-\delta_i} \le G_i(t,s) \le \frac{1}{1-\delta_i}, \;\; i = 1, 2, \cdots, n \tag{3.6}$$

and

$$\frac{1}{\hat{\delta}_j - 1} \le \hat{G}_j(t,s) \le \frac{\hat{\delta}_j}{\hat{\delta}_j - 1}, \;\; j = 1, 2, \cdots, m, \tag{3.7}$$

where $\delta_i = e^{-\theta_i \int_0^T r_i(z)\,dz}$, $\hat{\delta}_j = e^{\rho_j \int_0^T \hat{r}_j(z)\,dz}$.

Proof of Theorem 1.3

Let

$$X = \{x \in C(\mathbb{R}^{n+m}, \mathbb{R}) : x(t+T) = x(t)\}$$

endowed with the norm

$$\|x\| = \max_{1 \le i \le n+m} |x_i|_0,$$

where

$$|x_i|_0 = \sup_{t \in [0,T]} \{|x_i(t)|\}, \;\; i = 1, 2, \cdots, n+m.$$

Then X is a Banach space (3.6,3.7).

Let us define a cone P on X as

$$P = \{x = (u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_m) \in X : \\ u_i(t) \ge \delta_i |u_i|_0, v_j(t) \ge \frac{1}{\hat{\delta}_j} |v_j|_0, \; t \in [0, T]\}$$

for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$.

Proof of Theorem 1.3

Define an operator Φ on X by

$$(\Phi x)(t) = ((\Phi_1 x)(t), (\Phi_2 x)(t), \cdots, (\Phi_n x)(t), (\Psi_1 x)(t), (\Psi_2 x)(t), \cdots, (\Psi_m x)(t))^T \tag{3.8}$$

where

$$(\Phi_i x)(t) = \theta_i \int_t^{t+T} G_i(t,s) u_i(s) F_i(s, u(s), v(s))\, ds,\ i = 1, 2, \cdots, n \tag{3.9}$$

and

$$(\Psi_j x)(t) = \rho_j \int_t^{t+T} \hat{G}_j(t,s) v_j(s) \hat{F}_j(s, u(s), v(s))\, ds,\ j = 1, 2, \cdots, m. \tag{3.10}$$

Proof of Theorem 1.3

Set

$$\Pi = \max\left\{\frac{1}{\Pi_1}, \frac{1}{\Pi_2}, \cdots, \frac{1}{\Pi_n}, \frac{1}{\hat{\Pi}_1}, \frac{1}{\hat{\Pi}_2}, \cdots, \frac{1}{\hat{\Pi}_m}\right\},$$

and two positive constants r and R, with $0 < r < \Gamma$ and $R > \Pi$. Since $\Gamma_i > \Pi_i$ and $\hat{\Gamma}_j > \hat{\Pi}_j$, then we have $0 < r < \Gamma < \Pi < R$. Now, we consider the set

$$D = \{x \in P : r \leq \|x\| \leq R\}.$$

Using (3.6) and (3.7), we can show that $\Phi : D \to P$, compact and continuous.

First, suppose that $x \in D$ with $x = \mu\Phi x$, and $\mu \in (0, 1)$. We claim that $\|x\| \neq r$.

Proof of Theorem 1.3

For any $t \in [0, T]$, we have

$$
\begin{aligned}
|(\Phi_i x)(t)| =& \theta_i \int_t^{t+T} G_i(t,s) u_i(s) F_i(s, u(s), v(s))\, ds, \\
\leq & \frac{\theta_i}{1-\delta_i} \int_0^T u_i(s) F_i(s, u(s), v(s))\, ds, \\
=& \frac{\theta_i}{1-\delta_i} \int_0^T u_i(s) \Bigg[\sum_{k=1}^{n} a_{ik}(s) u_k(s - \tau_{ik}(s)) + \sum_{l=1}^{m} b_{il}(s) v_l(s - \sigma_{il}(s)) \\
& + \sum_{k=1}^{n} c_{ik}(s) \int_{-\infty}^{0} K_{ik}(\tau) u_k(\tau + s)\, d\tau + \sum_{l=1}^{m} d_{il}(s) \int_{-\infty}^{0} L_{il}(\tau) v_l(\tau + s)\, d\tau \Bigg]\, ds \\
\leq & \frac{\theta_i |u_i|_0}{1-\delta_i} \int_0^T \Bigg[\sum_{k=1}^{n} a_{ik}(s) |u_k|_0 + \sum_{l=1}^{m} b_{il}(s) |v_l|_0 \\
& + \sum_{k=1}^{n} c_{ik}(s) \int_{-\infty}^{0} K_{ik}(\tau) |u_k|_0\, d\tau + \sum_{l=1}^{m} d_{il}(s) \int_{-\infty}^{0} L_{il}(\tau) |v_l|_0\, d\tau \Bigg]\, ds \\
\leq & \Gamma_i \|x\|^2 = \frac{\Gamma_i}{\Gamma} \Gamma \|x\|^2 = \Gamma_i \Gamma \frac{r}{\Gamma} \|x\| < \|x\| \qquad (3.11)
\end{aligned}
$$

Proof of Theorem 1.3

$$
\begin{aligned}
|(\Psi_j x)(t)| =& \rho_j \int_t^{t+T} \hat{G}_j(t,s) v_j(s) \hat{F}_j(s, u(s), v(s))\, ds, \\
\leq& \frac{\rho_j \hat{\delta}_j}{\hat{\delta}_j - 1} \int_0^T v_j(s) \hat{F}_j(s, u(s), v(s))\, ds, \\
=& \frac{\rho_j \hat{\delta}_j}{\hat{\delta}_j - 1} \int_0^T v_j(s) \Bigg[\sum_{k=1}^{n} \hat{a}_{ik}(s) u_k(s - \tau_{ik}(s)) + \sum_{l=1}^{m} \hat{b}_{il}(s) v_l(s - \sigma_{il}(s)) \\
&+ \sum_{k=1}^{n} \hat{c}_{ik}(s) \int_{-\infty}^{0} \hat{K}_{ik}(\tau) u_k(\tau + s)\, d\tau + \sum_{l=1}^{m} \hat{d}_{il}(s) \int_{-\infty}^{0} \hat{L}_{il}(\tau) v_l(\tau + s)\, d\tau \Bigg] ds \\
\leq& \frac{\rho_j \hat{\delta}_j |v_j|_0}{\hat{\delta}_j - 1} \int_0^T \Bigg[\sum_{k=1}^{n} \hat{a}_{ik}(s) |u_k|_0 + \sum_{l=1}^{m} \hat{b}_{il}(s) |v_l|_0 \\
&+ \sum_{k=1}^{n} \hat{c}_{ik}(s) \int_{-\infty}^{0} \hat{K}_{ik}(\tau) |u_k|_0\, d\tau + \sum_{l=1}^{m} \hat{d}_{il}(s) \int_{-\infty}^{0} \hat{L}_{il}(\tau) |v_l|_0\, d\tau \Bigg] ds \\
\leq& \hat{\Gamma}_j \|x\|^2 = \frac{\hat{\Gamma}_j}{\Gamma} \Gamma \|x\|^2 = \hat{\Gamma}_j \Gamma \frac{r}{\Gamma} \|x\| < \|x\|. \qquad (3.12)
\end{aligned}
$$

Proof of Theorem 1.3

From the inequalities (3.11) and (3.12), we obtain
$r = \|x\| = \mu\|\Phi x\| < \|x\| = r$, which is a contradiction.

Hence $\|x\| \neq r$.

Thus, the condition $(\mathbf{b})$ of Theorem 1.1 is satisfied.

Proof of Theorem 1.3

Next, suppose that $x \in D$ and $x = \mu\Phi x$ with $\mu > 1$. We claim that $\|x\| \neq R$. If $\|x\| = R$, then for any $t \in [0, T]$, we have

$$\begin{aligned}
|(\Phi_i x)(t)| =& \theta_i \int_t^{t+T} G_i(t,s) u_i(s) F_i(s, u(s), v(s))\, ds, \\
\geq& \frac{\theta_i \delta_i}{1-\delta_i} \int_0^T u_i(s) F_i(s, u(s), v(s))\, ds, \\
=& \frac{\theta_i \delta_i}{1-\delta_i} \int_0^T u_i(s) \Bigg[\sum_{k=1}^n a_{ik}(s) u_k(s - \tau_{ik}(s)) + \sum_{l=1}^m b_{il}(s) v_l(s - \sigma_{il}(s)) \\
& + \sum_{k=1}^n c_{ik}(s) \int_{-\infty}^0 K_{ik}(\tau) u_k(\tau + s)\, d\tau + \sum_{l=1}^m d_{il}(s) \int_{-\infty}^0 L_{il}(\tau) v_l(\tau + s)\, d\tau \Bigg] \, ds \\
\geq& \frac{\theta_i \delta_i^2 |u_i|_0}{1-\delta_i} \int_0^T \Bigg[\sum_{k=1}^n a_{ik}(s) |u_k|_0 + \sum_{l=1}^m b_{il}(s) |v_l|_0 \\
& + \sum_{k=1}^n c_{ik}(s) \int_{-\infty}^0 K_{ik}(\tau) |u_k|_0\, d\tau + \sum_{l=1}^m d_{il}(s) \int_{-\infty}^0 L_{il}(\tau) |v_l|_0\, d\tau \Bigg] \, ds \\
\geq& \Pi_i \|x\|^2 = \frac{\Pi_i}{\Pi} \Pi \|x\|^2 = \Pi_i \Pi \frac{R}{\Pi} \|x\| > \|x\| \qquad (3.13)
\end{aligned}$$

Proof of Theorem 1.3

$$
\begin{aligned}
|(\Psi_j x)(t)| =& \rho_j \int_t^{t+T} \hat{G}_j(t,s) v_j(s) \hat{F}_j(s, u(s), v(s))\, ds, \\
\geq & \frac{\rho_j}{\hat{\delta}_j - 1} \int_0^T v_v(s) \hat{F}_j(s, u(s), v(s))\, ds, \\
=& \frac{\rho_j}{\hat{\delta}_j - 1} \int_0^T v_j(s) \Bigg[\sum_{k=1}^n \hat{a}_{ik}(s) u_k(s - \tau_{ik}(s)) + \sum_{l=1}^m \hat{b}_{il}(s) v_l(s - \sigma_{il}(s)) \\
& + \sum_{k=1}^n \hat{c}_{ik}(s) \int_{-\infty}^0 \hat{K}_{ik}(\tau) u_k(\tau + s)\, d\tau + \sum_{l=1}^m \hat{d}_{il}(s) \int_{-\infty}^0 \hat{L}_{il}(\tau) v_l(\tau + s)\, d\tau \Bigg]\, ds \\
\geq & \frac{\rho_j |v_j|_0}{\hat{\delta}_j(\hat{\delta}_j - 1)} \int_0^T \Bigg[\sum_{k=1}^n \hat{a}_{ik}(s) |u_k|_0 + \sum_{l=1}^m \hat{b}_{il}(s) |v_l|_0 \\
& + \sum_{k=1}^n \hat{c}_{ik}(s) \int_{-\infty}^0 \hat{K}_{ik}(\tau) |u_k|_0\, d\tau + \sum_{l=1}^m \hat{d}_{il}(s) \int_{-\infty}^0 \hat{L}_{il}(\tau) |v_l|_0\, d\tau \Bigg]\, ds \\
\geq & \hat{\Pi}_j \|x\|^2 = \frac{\hat{\Pi}_j}{\Pi} \Pi \|x\|^2 = \hat{\Pi}_j \Pi \frac{R}{\Pi} \|x\| > \|x\|. \qquad (3.14)
\end{aligned}
$$

Proof of Theorem 1.3

By (3.13) and (3.14), we have $R = \|x\| = \mu\|\Phi x\| > \|x\| = R$, which is a contradiction. Hence $\|x\| \neq R$.

Thus, the condition $(\mathbf{a})$ of Theorem 1.1 is satisfied.

Furthermore, the conditions (3.13) and (3.14) implies that $\inf_{\|x\|=R} \|\Phi x\| = R \neq 0$ holds.

This proves the condition $(\mathbf{c})$ of Theorem 1.1.

Hence, by Theorem 1.1, the system (3.2) has a positive T-periodic solution. Consequently, the system (1.2) has a positive T-periodic solution. □

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Thank you