National Seminar on "Real-World Applications *of Mathematics and Statistics*"

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Introduction

- In many cases it is of interest to model the evolution of some system over time.
- There are two distinct cases.
- One can think of time as a continuous variable, or one can think of time as a discrete variable.
- The first case often leads to differential equations.
- If we consider a time period T and observe (or measure) the system at times $t = kT$, $k \in N_0$, the result is a sequence $x_0, x_1, x_2, \ldots$.
- In some cases these values are obtained from a function f, which is defined for all $t \geq 0$.
- In this case $x_k = f(kT)$ and this method of obtaining the values is called periodic sampling.
- One models the system using a **difference equation**, or what is sometimes called a **recurrence relation**.

Introduction

- Difference equations arises in many fields of science, for example:
- In control engineering, the radar tracking devices receive discrete pulses from the target which is being tracked.
- In electrical networks, the electrical signals are measured in discrete time pulses
- Difference equations also arises in theory of probability, statistical problems and many other fields.
- In fact, difference equations are essential for systems with discrete or digital data.

Example

Consider a plane that has lying in it k nonparallel lines. Into how many separate compartments will the plane be divided if not more than two lines intersect in the same point?

Solution. Let N_k be the number of compartments.

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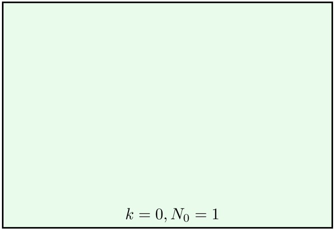

When $k = 0$, there are no lines and hence the plane is undivided and hence one compartment.

Example

Consider a plane that has lying in it k nonparallel lines. Into how many separate compartments will the plane be divided if not more than two lines intersect in the same point?

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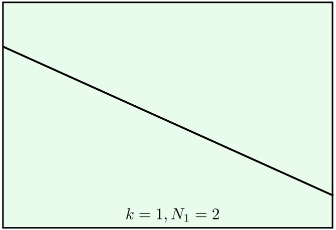
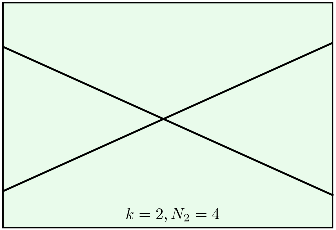


When $k = 1$, there is one line, which divide the previous one compartment into two compartments and thus making total two compartments.

Example

Consider a plane that has lying in it k nonparallel lines. Into how many separate compartments will the plane be divided if not more than two lines intersect in the same point?

Solution. Let N_k be the number of compartments.

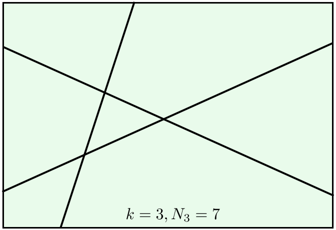


When $k = 2$, the second line, cut the previous one line at one point and divide the previous two compartments into two compartments and thus making total four compartments.

Example

Consider a plane that has lying in it k nonparallel lines. Into how many separate compartments will the plane be divided if not more than two lines intersect in the same point?

Solution. Let N_k be the number of compartments.

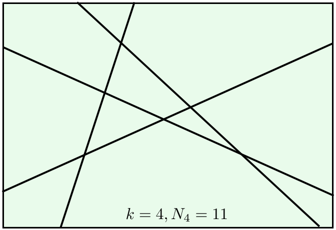


When $k = 3$, the third line, cut the previous two lines at two points and divide the previous three compartments into two compartments and thus making total seven compartments.

Example

Consider a plane that has lying in it k nonparallel lines. Into how many separate compartments will the plane be divided if not more than two lines intersect in the same point?

Solution. Let N_k be the number of compartments.



When $k = 4$, the fourth line, cut the previous three lines at three points and divide the previous four compartments into two compartments and thus making total eleven compartments.

Example

Consider a plane that has lying in it k nonparallel lines. Into how many separate compartments will the plane be divided if not more than two lines intersect in the same point?

Solution. Let N_k be the number of compartments.

So, generalizing, we have that the $(k+1)^{\text{th}}$ line will be cut by k previous lines in k points and, consequently, divides each of the $k+1$ prior existing compartments into two. This gives the difference equation

$$N_{k+1} = N_k + (k+1).$$

Difference Equation

An ordinary **difference equation** is a relation, of the form

$$y_{k+n} = F(k, y_{k+n-1}, y_{k+n-2}, \ldots, y_k) \tag{1}$$

between the differences of an unknown function at one or more general values of the argument.

Order of a difference equation

Order of a difference equation is the difference between the highest and the lowest indices that appear in the equation.

Remark

1. The expression given by the equation (1) is an n^{th}-order difference equation if and only if the term y_k appears in the function F on the right-hand side.
2. Shifts in the labeling of the indices do not changed the order of a difference equation. For example, for r integer,

$$y_{k+n+r} = F(k+r, y_{k+n+r-1}, y_{k+n+r-2}, \ldots, y_{k+r}) \tag{2}$$

is the n^{th}-order difference equation, which is equivalent to equation (1).

Linear Difference Equation

A difference equation is linear if it can be put in the form

$$y_{k+n}+a_1(k)y_{k+n-1}+a_2(k)y_{k+n-2}+\ldots+a_{n-1}y_{k+1}+a_n(k)y_k=R_k, \tag{3}$$

where $a_i(k), i=1,2,\ldots,n$ and R_k are given functions of k.

A difference equation is nonlinear if it is not linear.

Example (Drug Delivery)

A drug is administered once every four hours. Let $D(k)$ be the amount of the drug in the blood system at the k^{th} interval. The body eliminates a certain fraction p of the drug during each time interval. If the amount administered is D_0, find $D(k)$.

Example (Drug Delivery)

Solution: We first must create an equation to solve.
Since the amount of drug in the patient's system at time $(k+1)$ is equal to the amount at time k minus the fraction p that has been eliminated from the body, plus the new dosage D_0, we arrive at the following equation:

$$D(k+1) = (1-p)D(k) + D_0.$$

We can solve the above equation, arriving at

$$D(k) = \left[D_0 - \frac{D_0}{p}\right](1-p)^k + \frac{D_0}{p}.$$

Also,

$$\lim_{k\to\infty} D(k) = \frac{D_0}{p}.$$

Example (Applications to Economics)

- Here we study the pricing of a certain commodity. Let $S(k)$ be the number of units supplied in period k, $D(k)$ the number of units demanded in period k, and $p(k)$ the price per unit in period k.
- For simplicity, we assume that $D(k)$ depends only linearly on $p(k)$ and is denoted by

$$D(k) = -m_d p(k) + b_d, \qquad m_d > 0, \qquad b_d > 0. \tag{4}$$

- This equation is referred to as the price-demand curve. The constant m_d represents the sensitivity of consumers towards the price.
- The slop of the demand curve is negative because an increase of one unit in price produces a decrease of m_d units in demand.

Example (Applications to Economics)

- We also assume that the price-supply curve relates the supply in any period to the price one period before, i.e.,

$$S(k+1) = m_s p(k) + b_s, \qquad m_s > 0, \qquad b_s > 0. \quad (5)$$

- The constant m_s is the sensitivity of suppliers to price.
- An increase of one unit in price causes an increase of m_s units in supply, thus creating a positive slope for that price-supply curve.

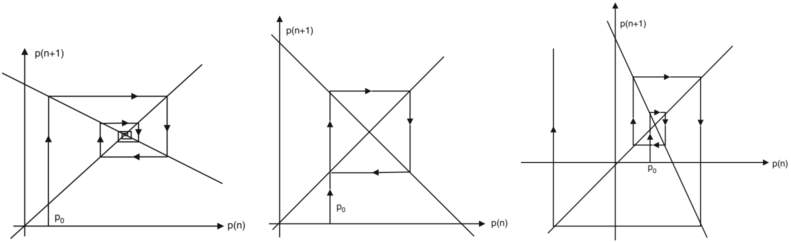
Example (Applications to Economics)

- A third assumption we make here is that the market price is the price at which the quantity demanded and the quantity supplied are equal, that is, at which $D(k+1) = S(k+1)$. Thus
$$-m_d p(k+1) + b_d = m_s p(k) + b_s,$$
or
$$p(k+1) = Ap(k) + B = f(p(k)), \tag{6}$$
where
$$A = -\frac{m_s}{m_d}, \qquad B = \frac{b_d - b_s}{m_d}. \tag{7}$$
This equation is a first-order linear difference equation.

Example (Applications to Economics)

An explicit solution of this difference equation with $p(0) = p_0$ is given by

$$p(k) = \left(p_0 - \frac{B}{1-A}\right) A^k + \frac{B}{1-A}$$



Example (Euler Scheme)

- Suppose the following differential equation:

$$\frac{dy}{dt} = f(y, t). \tag{8}$$

where $f(y, t)$ is a given function of y and t, which cannot be integrated in closed form in terms of the elementary functions.

- We now proceed to construct the a numerical scheme to determine the numerical solution.

Example (Euler Scheme)

- First, construct a lattice $t_k = (\Delta t)k$, where Δt is a fixed t interval and k is the set of integers.
- Secondly, replace the derivative by the approximation,

$$\frac{dy(t)}{dt} \approx \frac{y(t+\Delta t) - y(t)}{\Delta t} = \frac{y_{k+1} - y_k}{\Delta t}$$

  where y_k is the approximation to the exact solution of the equation at $t = t_k$ i.e., $y_k \approx y(t_k)$.
- Also, the right-hand side of equation becomes $f(y, t) \approx f(y_k, (\Delta t)k)$.

Example (Euler Scheme)

- Putting all of this together, we have

$$\frac{y_{k+1} - y_k}{\Delta t} = f(y_k, (\Delta t)k).$$

- If y_0 is specified, then y_k for $k = 1, 2, 3, \ldots,$ can be determined.
- This elementary method is called forward-Euler scheme.

Example (Power series solutions)

- Let us determine a power-series solution $y(x) = \sum_{k=0}^{\infty} C_k x^k$ to the differential equation

$$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = 0, \tag{9}$$

where the coefficients, C_k, are to be found.

Example (Power series solutions)

- We have

$$x\frac{dy}{dx} = x\sum_{k=0}^{\infty} kC_k x^{k-1} = \sum_{k=2}^{\infty}(k-2)C_{k-2}x^{k-2} \tag{10}$$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} k(k-1)C_k x^{k-2} = \sum_{k=2}^{\infty} k(k-1)C_k x^{k-2} \tag{11}$$

and

$$y = \sum_{k=0}^{\infty} C_k x^k = \sum_{k=2}^{\infty} C_{k-2}x^{k-2} \tag{12}$$

Example (Power series solutions)

- Substituting the above equations in the given differential equation, we obtain

$$\sum_{k=2}^{\infty} \left[k(k-1)C_k + 3(k-1)C_{k-2}\right] x^{k-2} = 0. \tag{13}$$

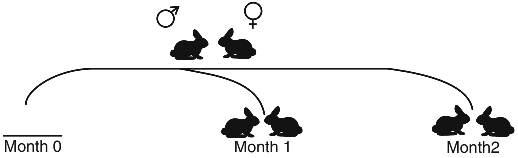
- Equation each coefficient to zero gives the following recursion relation, which the C_k must satisfy:

$$kC_k + 3C_{k-2} = 0. \tag{14}$$

Example (Fibonacci Sequence - Rabbit Problem)

This problem first appeared in 1202, in Liber abaci, a book about the abacus, written by the famous Italian mathematician Leonardo di Pisa, better known as Fibonacci. The problem may be stated as follows:

How many pairs of rabbits will there be after one year if starting with one pair of mature rabbits, if each pair of rabbits gives birth to a new pair each month starting when it reaches its maturity age of two months? (See Figure below)



Example (Fibonacci Sequence - Rabbit Problem)

Table below shows the number of pairs of rabbits at the end of each month.
The first pair has offspring at the end of the first month, and thus we have two pairs.
At the end of the second month only the first pair has offspring, and thus we have three pairs.
At the end of the third month, the first and second pairs will have offspring, and hence we have five pairs. Continuing this procedure, we arrive at Table.

Month	0	1	2	3	4	5	6	7	8	9	10	11	12
Pairs	1	2	3	5	8	13	21	34	55	89	144	233	377

Example (Fibonacci Sequence - Rabbit Problem)

If $F(k)$ is the number of pairs of rabbits at the end of k months, then the recurrence relation that represents this model is given by the second-order linear difference equation

$$F(k+2) = F(k+1) + F(k), \quad F(0) = 1, \quad F(1) = 2, \quad 0 \leq 10.$$

This example is a spacial case of the Fibonacci sequence, given by

$$F(k+2) = F(k+1) + F(k), \quad F(0) = 0, \quad F(1) = 1, \quad n \geq 0. \quad (1)$$

The first 14 terms are given by $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233,$ and 377, as already noted in the rabbit problem.

Solution of a difference equation

- A **solution** of a difference equation is a function $\phi(k)$ that reduces the equation to an identity.
- The **general solution** of a difference equation is the solution in which the number of arbitrary constants is equal to the order of the difference equation.
- A **particular solution** of a difference equation is that solution which is obtained from the general solution by giving particular values to the constants.

Example

$$y_{k+1} - 3y_k + y_{k-1} = e^{-k} \qquad \text{(second order, linear)}$$
$$y_{k+1} = y_k^2 \qquad \text{(first order, nonlinear)}$$
$$y_{k+4} - y_k = k2^k \qquad \text{(fourth order, linear)}$$
$$y_{k+1} = y_k - (1/100)y_k^2 \qquad \text{(first order, nonlinear)}$$
$$y_{k+3} = \cos y_k \qquad \text{(third order, nonlinear)}$$
$$y_{k+2} + (3k-1)y_{k+1} - \frac{k}{k+1}y_k = 0 \qquad \text{(second order, linear)}$$

The first-order nonlinear equation

$$y_{k+1}^2 - y_k^2 = 1$$

has the solution $\phi(k) = \sqrt{k+c}$ where c is a constant.
This statement can be checked by substituting $\phi(k)$ into the left-hand side of the difference equation to obtain

$$(\sqrt{k+1+c}^2 - \sqrt{k+c}^2 = (k+1-c) - (k+c) = 1,$$

which is equal to the expression on the right-hand side.

The second-order linear difference equation

$$y_{k+1} - y_{k-1} = 0$$

has two solutions, $\phi_1(k) = (-1)^k$ and $\phi_2(k) = 1$.

Thoerem (Existence and Uniqueness Theorem)

Let

$$y_{k+n} = f(k, y_k, y_{k+1}, \ldots, y_{k+n-1}), \quad k = 0, 1, 2, 3, \ldots$$

be an n^{th}*-order difference equation where* f *is defined for each of its arguments. This equation has one and only one solution corresponding to each arbitrary selection of the* n *initial values* $y_0, y_1, \ldots, y_{n-1}$.

Proof.

If the values, $y_0, y_1, \ldots, y_{n-1}$ are given, then the difference equation with $k = 0$ uniquely specifies, y_n. Once y_n is known, the difference equation with $k = 1$ gives y_{n+1}. Proceeding in this way, all y_k, for $k \geq n$, can be determined. $\square$

Operators Δ and E

- In the theory of difference equations, more frequently, we use the operators Δ and E to denote the differences:
- The operator Δ (called as (first) difference operator) is defined as follows:

$$\Delta y_k = y_{k+1} - y_k.$$

- The second difference operator is defined as $\Delta^2 = \Delta \cdot \Delta$,

$$\Delta^2 y_k = \Delta(\Delta(y_k)) = \Delta(y_{k+1} - y_k) = y_{k+2} - 2y_{k+1} + y_k.$$

- In general,

$$\begin{aligned}\Delta^n y_k = & y_{k+n} - ny_{k+n-1} + \frac{n(n-1)}{2!}y_{k+n-2} + \dots \\ & + (-1)^i \frac{n(n-1)\dots(n-i+1)}{i!} y_{k+n-i} + \dots + (-1)^n y_k\end{aligned}$$

Operators Δ and E

- The operator, E, (called *shift operator*) is defined as

$$E^p = y_{k+p}.$$

- From the definition of Δ and E, we have

$$\Delta y_k = (E-1)y_k$$

  and that

$$\Delta \equiv E - 1 \text{ or } E \equiv \Delta + 1.$$

- Hence, we have

$$y_{k+n} = E^n y_k = (1+\Delta)^n y_k = \sum_{i=0}^{n} \binom{n}{i} \Delta^i y_k.$$

Definition

Let the functions $a_0(k), a_1(k), \ldots, a_n(k)$, and R_k be defined over a set of integers, $k_1 \leq k \leq k_2$, where k_1 and k_2 can be either finite or unbounded in magnitude. An equation of the form

$$a_0(k)y_{k+n} + a_1(k)y_{k+n-1} + \ldots + a_n(k)y_k = R_k$$

is said to be linear. This equation is of order n if and only if $a_0(k) \neq 0$, for any k. In this case dividing by $a_0(k)$ and relabeling the ratio of coefficient functions, we can write the general n^{th} order linear difference equation as follows:

$$y_{k+n} + a_1(k)y_{k+n-1} + \ldots + a_n(k)y_k = R_k. \tag{15}$$

Definition

The (15) is called *homogeneous* if R_k is identically zero for all k i.e.

$$y_{k+n} + a_1(k)y_{k+n-1} + \ldots + a_n(k)y_k = 0; \qquad (16)$$

otherwise, it is called an *inhomogeneous* equation.

Definition

If the functions $a_0(k), a_1(k), \ldots a_n(k)$ are constant then the (15) is said to Linear difference equations with constant coefficients.

Definition

A set of k linearly independent solutions of (16) is called a fundamental set of solutions.

Definition

The *Casoratian* $C(k)$ of the solutions $f_1(k), f_2(k), ..., f_n(k)$ is dined as

$$C(k) = \begin{vmatrix} f_1(k) & f_2(k) & \dots & f_n(k) \\ f_1(k+1) & f_2(k+1) & \dots & f_n(k+1) \\ \vdots & \vdots & & \\ f_1(k+n-1) & f_2(k+n-1) & \dots & f_n(k+n-1) \end{vmatrix}$$

Casoratian plays an important role in determining whether particular set of functions are linearly independent or dependent.

Thoerem

The functions $f_1(k), f_2(k), \dots, f_n(k)$ are n linearly dependent functions if and only if their Casoratian equals to for all k.

Fundamental Theorems for Homogeneous Equations

Thoerem

Let the functions, $a_1(k), a_2(k), \ldots, a_n(k)$ be defined for all k; let $a_n(k)$ be nonzero for all k; then there exist n linearly independent solutions $y_1(k), y_2(k), \ldots, y_n(k)$ of (16).

Thoerem

An n^{th}-order linear difference equation has n and only n linearly independent solutions.

Thoerem

The general solution of equation (16) is given by

$$y_k = c_1 y_1(k) + c_2 y_2(k) + \ldots + c_k y_n(k),$$

where c_i, $1 \leq i \leq n$, are n arbitrary constants and the $y_i(k)$, $1 \leq i \leq n$ ar a fundamental set of solutions.

We consider the n^{th}-order linear difference equations with constant coefficients,

$$y_{k+n} + a_1 y_{k+n-1} + \ldots + a_n y_k = R_k, \tag{17}$$

where a_i are given set of n constants, with $a_n \neq 0$, and R_k is a given function of k. If $R_k = 0$, then (17) is homogeneous:

$$y_{k+n} + a_1 y_{k+n-1} + \ldots + a_n y_k = 0; \tag{18}$$

for $R_k \neq 0$, equation (17) is inhomogeneous.

From, the previous slides, we have the homogeneous equation (18) has a fundamental set of solutions that consists of n linearly independent functions $y_k^{(1)}, y_k^{(2)}, \ldots, y_k^{(n)}$ and that the general solution is the linear combination

$$y_k^{(H)} = c_1 y_k^{(1)} + c_2 y_k^{(2)} + \ldots + c_n y_k^{(n)},$$

where c_i are n arbitrary constants.

The inhomogeneous equation (17) consists of a sum of homogeneous solution $y_k^{(H)}$ and a particular solution $y_k^{(P)}$ to equation (17),

$$y_k = y_k^{(H)} + y_k^{(P)}.$$

Using the shift operator E, we can write the equation (17) as

$$f(E)y_k = 0, \tag{19}$$

where

$$f(E) = E^n + a_1 E^{n-1} + \ldots + a_{n-1}E + a_n. \tag{20}$$

Definition

The characteristic equation associated with equation (17) or (19) is

$$f(r) = r^n + a_1 r^{n-1} + \ldots + a_{n-1}r + a_n = 0. \tag{21}$$

Thoerem

Let r_i be any solution to the characteristic equation of (21); then

$$y_k = r_i^k \tag{22}$$

is a solution to the homogeneous equation (18).

Thoerem

Assume the n roots of the characteristic equations are distinct; then a fundamental set of solution is $y_k^{(i)} = r_i^k, i = 1, 2, \ldots, n$ *and that the general solution to the homogeneous equation (18) is*

$$y_k = c_1 y_k^{(1)} + c_2 y_k^{(2)} + \ldots + c_n y_k^{(n)}, \tag{23}$$

where the n constants c_i *are arbitrary.*

Thoerem

Let the roots of the characteristic equation (21) of the homogeneous difference equation (18) be r_i with multiplicity m_i, $i = 1, 2, \ldots, l$, where $m_1 + m_2 + \ldots + m_l = n$. Then, the general solution of (21) is:

$$\begin{aligned} y_k =& r_1^k(A_1^{(1)} + A_2^{(1)}k + \ldots + A_{m_1}^{(1)}k^{m_1 - l}) \\ &+ r_2^k(A_1^{(2)} + A_2^{(2)}k + \ldots + A_{m_2}^{(2)}k^{m_2 - l}) \\ &+ \ldots \\ &+ r_{m_l}^k(A_1^{(l)} + A_2^{(l)}k + \ldots + A_{m_l}^{(l)}k^{m_l - l}). \end{aligned} \tag{24}$$

Thoerem

If the linear difference equation (18) has real coefficients, then any complex roots of the characteristic equation (21) must occur in complex conjugates pairs. Moreover, the corresponding fundamental solutions can be written in either of the two equivalent forms:

$$y_k^{(1)} = y_k^{(2)*} = r_1^k$$

or

$$\bar{y}_k^{(1)} = R^k \cos(k\theta), \bar{y}_k^{(2)} = R^k \sin(k\theta),$$

where the complex conjugate pair of roots are

$$r_1 = r_2^* = a + ib = Re^{i\theta}; \qquad R = \sqrt{a^2 + b^2}, \tan\theta = b/a.$$

In particular, for a homogeneous linear difference equation of order 2, i.e., $y_{k+2} + a_1 y_{k+1} + a_2 y_k = 0 (a_2 \neq 0)$, we have the following three situations.

Three Cases

Case 1: The characteristic roots r_1, r_2 are real and distinct. Then the general solution is

$$y_k = c_1 r_1^k + c_2 r_2^k.$$

Case 2: The characteristic roots r_1, r_2 are real and equal (say $r_1 = r_2 = r$). Then the general solution is

$$y_k = (c_1 + c_2 k) r^k.$$

Three Cases

Case 3: The characteristic roots are complex conjugates, say $r_{1,2} = a \pm ib = Re^{\pm i\theta}$, where $R = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. Then the general solution is

$$
\begin{aligned}
y_k &= c_1 \left(Re^{i\theta}\right)^k + c_2 \left(Re^{-i\theta}\right)^k \\
&= R^k \left[c_1(\cos k\theta + i \sin k\theta) + c_2(\cos k\theta - i \sin k\theta)\right] \\
&= R^k \left[(c_1 + c_2) \cos k\theta + (ic_1 - ic_2) \sin k\theta\right] \\
&= R^k \left[A_1 \cos k\theta + A_2 \sin k\theta\right]
\end{aligned}
$$

where A_1, A_2 are arbitrary constants.

Example

Find the general solution of

$$y_{k+2} + 5y_{k+1} + 6y_k = 0.$$

Solution

The characteristic equation for the given problem is:

$$r^2 + 5r + 6 = 0,$$

which has the roots $r_1 = -3$ and $r_2 = -2$. Therefore, the general solution is

$$y_k = c_1(-3)^k + c_2(-2)^k.$$

Example

Find the general solution of

$$y_{k+2} - 2y_{k+1} + y_k = 0.$$

Solution

The characteristic equation for the given problem is:

$$r^2 - 2r + 1 = 0 \text{ or } (r-1)^2 = 0,$$

which has the roots $r_1 = r_2 = 1$. Therefore, the general solution is

$$y_k = (c_1 + c_2 k)(1)^k = c_1 + c_2 k.$$

Example

Find the general solution of

$$y_{k+2} - 2y_{k+1} - 2y_k = 0.$$

Solution

The characteristic equation for the given problem is:

$$r^2 - 2r + 2 = 0,$$

which has the complex conjugate roots $r_{1,2} = 1 \pm i$. Thus $R = \sqrt{2}$ and $\theta = \tan^{-1} 1 = \frac{\pi}{4}$.
Therefore, the general solution is

$$y_k = (\sqrt{2})^k \left[A_1 \cos\left(k\frac{\pi}{4}\right) + A_2 \sin\left(k\frac{\pi}{4}\right) \right].$$

Example (Fibonacci Sequence - Revisited)

$$F(k+2) = F(k+1) + F(k), \quad F(0) = 0, \quad F(1) = 1, \quad k \geq 0. \quad (1)$$

The characteristic equation of (1) is

$$r^2 - r - 1 = 0.$$

Hence the characteristic roots are $r_1 = \alpha = \frac{1+\sqrt{5}}{2}$ and $r_2 = \beta = \frac{1-\sqrt{5}}{2}$.
The general solution of (1) is

$$F(k) = a_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + a_2 \left(\frac{1-\sqrt{5}}{2}\right)^n, \quad k \geq 1. \quad (2)$$

Example (Fibonacci Sequence - Revisited)

Using the initial values $F(1) = 1$ and $F(2) = 1$, one obtains

$$a_1 = \frac{1}{\sqrt{5}}, \quad a_2 = -\frac{1}{\sqrt{5}}.$$

Consequently,

$$F(k) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] = \frac{1}{\sqrt{5}}(\alpha^k - \beta^k). \quad (3)$$

It is interesting to note that $\lim_{k\to\infty} \frac{F(k+1)}{F(k)} = \alpha \approx 1.618$. This number is called the *golden mean/ratio*, which supposedly represents the ratio of the sides of a rectangle that is most pleasing to the eye.

Example (The Transmission of Information)

- Suppose that a signaling system has two signals s_1 and s_2 such as dots and dashes in telegraphy.
- Messages are transmitted by first encoding them into a string, or sequence, of these two signals.
- Let us suppose that s_1 requires exactly n_1 units of time, and s_2 requires exactly n_2 units of time, to be transmitted.
- Let M(n) be the number of possible message sequence of duration n and a signal of duration time n either ends with an s_1 signal or with an s_2 signal.

Example (The Transmission of Information)

- Now, if the message ends with s_1, the last signal must start at $n - n_1$.
- Hence there are $M(n - n_1)$ possible messages to which the message s_1 can be appended at the end.
- By a similar argument, there are $M(n - n_2)$ possible messages to which the message s_2 can be appended at the end.
- Consequently, the total number of messages $x(n)$ of duration n may be given by

$$M(n) = M(n - n_1) + M(n - n_2).$$

Example (The Transmission of Information)

- If $n_1 \geq n_2$ then we obtain a difference equation of n_1th-order

$$M(n+n_1) - M(n+n_1-n_2) - M(n) = 0.$$

- If $n_2 \geq n_1$ then we obtain a difference equation of n_2th-order

$$M(n+n_2) - M(n+n_2-n_1) - M(n) = 0.$$

- An interesting special case is that in which $n_1 = 1$ and $n_2 = 2$. In this case we have

$$M(n+2) - M(n+1) - M(n) = 0$$

which is nothing but our *Fibonacci Sequence*.

Example (The Transmission of Information)

- The general solution is

$$M(n) = a_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + a_2 \left(\frac{1-\sqrt{5}}{2}\right)^n, \quad n = 0, 1, 2, \ldots$$

- To find a_1, a_2 let us take $M(0) = 0, M(1) = 1$, this yields $a_1 = 1/\sqrt{5}, a_2 = -1/\sqrt{5}$.
- So, we have

$$M(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n, \quad n = 0, 1, 2, \ldots$$

Example (The Transmission of Information)

- In information theory, the capacity of C of the channel is defined as
$$C = \lim_{n\to\infty} \frac{\log_2 M(n)}{n}$$
- So,

$$\begin{aligned} C &= \lim_{n\to\infty} \frac{\log_2 \frac{1}{\sqrt{5}}}{n} + \lim_{n\to\infty} \frac{1}{n} \log_2 \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \\ &= \log_2 \left(\frac{1+\sqrt{5}}{2} \right) \\ &\approx 0.7 \end{aligned}$$

We now turn to a technique for obtaining solutions to the n^{th}-order linear inhomogeneous difference equations with constant coefficients,

$$y_{k+n} + a_1 y_{k+n-1} + \ldots + a_n y_k = R_k, \qquad a_n \neq 0, \tag{25}$$

where R_k is a linear combination of terms each having one of the forms

$$a^k, \quad e^{bk}, \quad \sin(ck) \quad \cos(ck), \quad k^l,$$

where a, b, and c are constants and l is a non-negative integer. We also include products of these forms; for example,

$$a^k \cos(ck), \quad k^l e^{bk}, \quad a^k k^l \cos(ck), \quad etc.$$

To proceed, we first need some definitions.

Definition

A *family* of a term R_k is the set of all functions of which R_k and $E^m R_k$, for $m = 1, 2, 3, \ldots,$ are linear combinations.

Definition

A *finite family* is a family that contains only a finite number of functions.

For example, if $R_k = a^k$, then $E^m a^k = a^m a^k$, $m = 1, 2, 3, \ldots$, and the family a^k contains only one member, namely, a^k. We denote this family by $\{a^k\}$.

If $R_k = k^l$, then $E^m k^l = (k+m)^l$, which can be expressed as a linear combination of $1, k, k^2, \ldots, k^l$; thus, the family of $E^m k^l$ is the set $\{1, k, k^2, \ldots, k^l\}$.

If $R_k = \cos(ck)$ or $\sin(ck)$, then the families are $\{\cos(ck), \sin(ck)\}$.

Finally, note that for the case R_k is a product, the family consists of all possible products of distinct members of the individual term families. For example, the term $R_k = k^l a^k$ has the finite family $\{a^k, ka^k, k^2 a^k, \ldots, k^l a^k\}$.
Likewise, the term $R_k = k^l \cos(ck)$ has the finite family $\{\cos(ck), k\cos(ck), \ldots, k^l \cos(ck), \sin(ck), k\sin(ck), \ldots, k^l \sin(ck)\}$.

Procedure for obtaining particular solutions to the inhomogeneous equation (25)

(i) Construct the family of R_k.

(ii) If the family contains no terms of the homogeneous solution, then write the particular solution $y_k^{(P)}$ as a linear combination of the members of that family. Determine the constants of combinations such that the inhomogeneous difference equation is identically satisfied.

(iii) If the family contains terms of the homogeneous solution, then multiply each member of the family by the smallest integral power of k for which all such terms are removed. The particular solution $y_k^{(P)}$ can then be written as a linear combination of the members of this modified family. Again, determine the constants of combination such that that the inhomogeneous difference equation is identically satisfied.

Example (A)

The second-order difference equation

$$y_{k+2} - 5y_{k+1} + 6y_k = 2 + 4k$$

has the characteristic equation $r^2 - 5r + 6 = (r-3)(r-2) = 0$, with roots $r_1 = 3$ and $r_2 = 2$. Therefore, the homogeneous solution is

$$y_k^{(H)} = c_1 3^k + c_2 2^k,$$

where c_1 and c_2 are arbitrary constants. The right-hand side of the difference equation is $2 + 4k$. Note that, the 2 has the family that consists of only one member $\{1\}$, while $4k$ has the family $\{1, k\}$. Therefore, the combined family is $\{1, k\}$. Since, neither member of the combined family occurs in the homogeneous solution, we write the particular solution as the following linear combination:

$$y_k^{(P)} = A + Bk,$$

where constants A and B are to be determined.

Example (A)

Substituting the above into the given difference, we obtain

$$A + B(k+2) - 5A - 5B(k+1) + 6A + 6Bk = 2 + 4k.$$

Upon setting coefficients of the k^0 and k^1 terms equal to zero, we obtain

$$2A - 3B = 2, 2B = 4.$$

Therefore,

$$A = 4, B - 2,$$

and the particular solution is

$$y_k^{(P)} = 4 + 2k.$$

The general solution to the given difference equation is:

$$y_k = c_1 3^k + c_2 2^k + 4 + 2k.$$

Example (B)

Consider the difference equation

$$y_{k+2} - 6y_{k+1} + 8y_k = 2 + 3k^2 - 5 \cdot 3^k.$$

The characteristic equation is: $r^2 - 6r + 8 = (r-2)(r-4) = 0$, which leads to the following solution of the homogeneous equation:

$$y_k^{(H)} = c_1 2^k + c_2 4^k,$$

where c_1 and c_2 are arbitrary constants. The families of the terms in R_k are

$$2 \to \{1\}; k^2 \to \{1, k, k^2\}; 3^k \to \{3^k\};$$

So the combined family is $\{1, k, k^2, 3^k\}$.

Example (B)

No member of this family occur in the homogeneous solution. Therefore, the particular solution takes the form

$$y_k^{(P)} = A + Bk + Ck^2 + D3^k,$$

where A, B, C, and D are constants to be determined. Substituting this in the given difference equation and simplifying the resulting expression gives

$$(3A - 4B - 2C) + (3B - 8C)k + 3ck^2 - D3^k = 2 + 3k^2 - 5 \cdot 3^k.$$

Equating the coefficients of the linearly independent terms on both sides to zero gives

$$3A - 4B - 2C = 2; 3B - 8C = 0; 3C = 3; D = 5.$$

Example (B)

Solving the above equations, we obtain

$$A = 44/9; B = 8/3; C = 1; D = 5,$$

and that the particular solution is

$$y_k^{(P)} = \frac{44}{9} + \frac{8}{3}k + k^2 + 5 \cdot 3^k,$$

and the general solution to the given difference is

$$y_k = c_1 2^k + c_2 4^k + \frac{44}{9} + \frac{8}{3}k + k^2 + 5 \cdot 3^k.$$

Example (C)

The equation

$$y_{k+2} - 4y_{k+1} + 3y_k = k4^k$$

has the homogeneous solution

$$y_k^H = c_1 + c_2 3^k,$$

where c_1 and c_2 are arbitrary constants. The family of $R_k = k4^k$ is $\{4^k, k4^k\}$ and does not contain a term that appears in the homogeneous solution. Therefore, the particular solution is of the form

$$y_k^{(P)} = (A + Bk)4^k,$$

where A and B can be determined by substituting this equation in the given difference equation;

Example (C)

doing this gives

$$(3A + 16B)4^k + (3B)k4^k = k4^k$$

and

$$3A + 16B = 0, 3B = 1; \text{ which implies } A = -16/9, B = 1/3.$$

So, the particular solution is

$$y_k^{(P)} = -\frac{16}{9}4^k + \frac{1}{3}k4^k$$

and the general solution is $y_l = c_1 + c_2 3^k - \frac{16}{9}4^k + \frac{1}{3}k4^k$.

Example (D)

Consider the third-order difference equation

$$y_{k+3} - 7y_{k+2} + 16y_{k+1} - 12y_k = k2^k.$$

Its characteristic equation is:

$$r^3 - 7r^2 + 16r - 12 = (r-2)^2(r-3) = 0,$$

and the corresponding homogeneous solution is:

$$y_k^{(H)} = (c_1 + c_2 k)2^k + c_3 3^k,$$

where c_1, c_2 and c_3 are arbitrary constants.

Example (D)

The family of $R_k = k2^k$ is $\{2^k, k2^k\}$ and both members of this family occur in the homogeneous solution; therefore, we must multiply the family by k^2 to obtain a new family that does not contain any function that appear in the homogeneous solution. The new family is $\{k^2 2^k, k^3 2^k\}$. Thus, the particular solution is

$$y_k^{(P)} = (Ak^2 + Bk^3)2^k,$$

where A and B are to be determined. The substitution of this equation into the difference equation, gives

$$2^k(-8A + 24B) + k2^k(-24B) = k2^k,$$

from which we obtain $A = -1/8$; $B = -1/24$.

Example (D)

Therefore, the particular solution is

$$y_k^{(P)} = -\frac{1}{24}(3+k)k^2 2^k,$$

and the general solution is

$$y_k = (c_1 + c_2 k)2^k + c_3 3^k - \frac{1}{24}(3+k)k^2 2^k.$$

Example (E)

Consider the second order difference equation

$$y(k+2)+4y(k)=8(2^k)\cos\left(\frac{k\pi}{2}\right).$$

The characteristic equation of the homogeneous equation is

$$r^2+4=0.$$

The characteristic roots are $r_1=2i, \quad r_2=-2i$. Thus $R=2, \theta=\pi/2$, and

$$y_k^{(H)}=2^k\left(c_1\cos\left(\frac{k\pi}{2}\right)+c_2\sin\left(\frac{k\pi}{2}\right)\right).$$

Example (E)

Notice that the family of $R_k = 8(2^k)\cos\cos\left(\frac{k\pi}{2}\right)$ is $\{(2^k)\cos\left(\frac{k\pi}{2}\right), (2^k)\sin\left(\frac{k\pi}{2}\right)\}$, both of the members belongs to the homogeneous solution. So, we assume

$$y_k^{(P)} = 2^k\left(ak\cos\left(\frac{k\pi}{2}\right) + bk\sin\left(k\pi 2\right)\right).$$

Substituting $y_k^{(P)}$ into given difference equation gives

$$2^{k+2}\left[a(k+2)\cos\left(\frac{k\pi}{2}+\pi\right) + b(k+2)\sin\left(\frac{k\pi}{2}+\pi\right)\right] + (4)2^k\left[ak\cos\left(\frac{k\pi}{2}\right) + bk\sin\left(\frac{k\pi}{2}\right)\right] = 8(2^k)\cos\left(\frac{k\pi}{2}\right).$$

Example (E)

Replacing $\cos((k\pi)/2+\pi)$ by $-\cos((k\pi)/2)$, and $\sin((k\pi)/2+\pi)$ by $-\sin((k\pi)/2)$ and then comparing the coefficients of the cosine terms leads us to $a=-1$. Then by comparing the coefficients of the sine terms, we realize that $b=0$.

By substituting these values back into $y_k^{(P)}$, we have that

$$y_k^{(P)} = -2^k k\cos\left(\frac{k\pi}{2}\right),$$

and the general solution is

$$y_k = 2^k\left(c_1\cos\left(\frac{k\pi}{2}\right) + c_2\sin\left(\frac{k\pi}{2}\right) - k\cos\left(\frac{k\pi}{2}\right)\right).$$

Example (The Logistic Equation)

- Let $y(k)$ be the size of a population at time k.
- If μ is the rate of growth of the population from one generation to another, then we may consider a mathematical model in the form

$$y(k+1) = \mu y(k), \qquad \mu > 0. \tag{26}$$

- If the initial population is given by $y(0) = y_0$, then by simple iteration we find that

$$y(k) = \mu^k y_0$$

is the solution of (1) .

Example (The Logistic Equation)

- If $\mu > 1$, then $y(k)$ increases indefinitely, and $\lim_{k\to\infty} y(k) = \infty$.
- If $\mu = 1$, then $y(k) = y_0$ for all $k > 0$, which means that the size of the population is constant for the indefinite future.
- However, for $\mu < 1$, we have $\lim_{k\to\infty} y(k) = 0$, and the population eventually becomes extinct.
- For most biological species, however, none of the above cases is valid as the population increases until it reaches a certain upper limit.
- Then, due to the limitations of available resources, the creature will become testy and engage in competition for those limited resources.

Example (The Logistic Equation)

- This competition is proportional to the number of squabbles among them, given by $y^2(k)$.
- A more reasonable model would allow b, the proportionality constant, to be greater than 0,

$$y(k+1) = \mu y(k) - by^2(k). \tag{27}$$

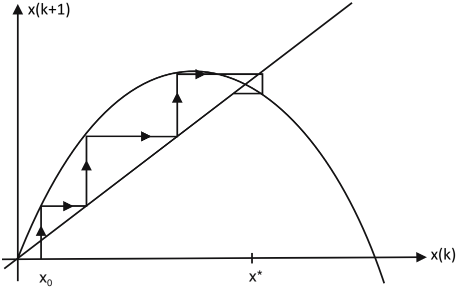
If in (27), we let $x(k) = \frac{b}{\mu} y(k)$, we obtain

$$x(k+1) = \mu x(k)(1 - x(k)) \equiv f(x(k)). \tag{28}$$

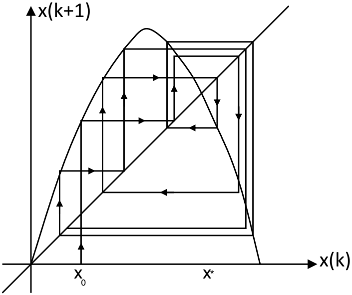
- This equation is the simplest nonlinear first-order difference equation, commonly referred to as the (discrete) logistic equation.

Example (The Logistic Equation)

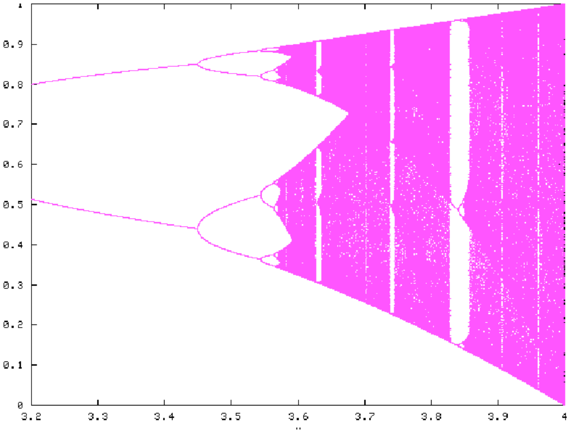
- However, a closed-form solution of (28) is not available (except for certain values of μ).
- In spite of its simplicity, this equation exhibits rather rich and complicated dynamics.
- To find the equilibrium points of (28) we let $f(x^*) = \mu x^*(1 - x^*) = x^*$. Thus, we pinpoint two equilibrium points: $x^* = 0$ and $x^* = (\mu - 1)/\mu$.



Stair step diagram of $(x(k), x(k+1))$ when $\mu = 2.5$ and $x(0) = 0.1$. In this case, we have two equilibrium points, one $x^* = 0$, which is unstable, and the other $x^* = 0.6$, which is asymptotically stable.



Stair step diagram of $(x(k), x(k+1))$ in which one interior equilibrium $x^* = 0$ is unstable and the other interior equilibrium is unstable.



Bifurcation diagram of Logistic Equation w.r.t. parameter μ.

Consider an nth-order linear homogeneous differential equation having constant coefficients,

$$D^n y(x) + a_1 D^{n-1} y(x) + \ldots + a_n y(x) = 0, \tag{29}$$

where $D \equiv d/dx$ is the differentiation operator, the a_i, $i = 1, 2, \ldots, n$, are given constants, and $a_n \neq 0$. Associated with this differential equation is the following difference equation:

$$y_{k+1} + a_1 y_{k+n-1} + \ldots + a_n y_k = 0. \tag{30}$$

Thoerem

Let

$$y(x) = \sum_{i=1}^{l} \left(\sum_{j=0}^{n_i-1} c_{i,j+1} x^j \right) e^{r_i x} + \sum_{j=(n_1+\ldots+n_l)+1}^{n} c_j e^{r_j x} \tag{31}$$

be the general solution of equation(1), where $c_{i,j+1}$ *and* c_j *are arbitrary constants;* $n_i \geq 1$*,*$i = 1, 2, \ldots, l$*, with* $n_1 + n_2 + \ldots + n_l \leq n$*; and where the characteristic equation*

$$r^n + a_1 r^{n-1} + \ldots + a_n = 0 \tag{32}$$

has roots r_i *with multiplicity* n_i, $i = 1, 2, \ldots, l$*, and the simple roots* r_j*.*
Let y_k *be the general solution of equation (2). Then*

$$y_k = D^k y(x)\Big|_{x=0}, \tag{33}$$

and

$$y_k = \sum_{i=1}^{i} \left(c_{i1} + \sum_{m=1}^{n_i-1} \gamma_i, m^{k^m} \right) r_i^k + \sum_{j=(n_1+\ldots+n_l)+1}^{n} c_j r_i^k, \tag{34}$$

where the $\gamma_{i,m}$ *are arbitrary constants.*

Example. The second-order differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \tag{35}$$

has the general solution

$$y(x) = c_1 e^x + c_2 e^{2x}, \tag{36}$$

where c_1 and c_2 are arbitrary constants. The difference equation associated with this differential equation is

$$y_{k+2} - 3y_{k+1} + 2y_k = 0. \tag{37}$$

Its general solution is

$$y_k = A + B2^k, \tag{38}$$

since the characteristic equation $r^2 - 3r + 2 = 0$ has roots $r_1 = 1$ and $r_2 = 2$; A and B are arbitrary constants. We now show how the result given by equation (38) can be obtained from equation (36). Let us calculate $D^k y(x)$;it is

$$D^k y(x) = \frac{d^k}{dx^k}(c_1 e^x + c_2 e^{2x}) = c_1 e^x + c_2 2^k e^{2x}. \tag{39}$$

Therefore,

$$y_k = D^k y(x)\Big|_{x=0} = c_1 + c_2 2^k, \tag{40}$$

which is the same as equation (38) except for the labeling of the arbitrary constants.

Example. The differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0 \tag{41}$$

has the general solution

$$y(x) = (c_1 + c_2 x)e^x = c_1 e^x + c_2 x e^x. \tag{42}$$

The associated difference equation is

$$y_{k+2} - 2y_{k+1} + y_k = 0. \tag{43}$$

From equation (42) we obtain

$$\begin{aligned} D_k y(x) &= \frac{d^k}{dx^k}(c_1 e^x + c_2 x e^x) \\ &= c_1 e^x + c_2(xe^x + ke^x), \end{aligned} \tag{44}$$

where the expression in parentheses on the right-hand side of equation (44) was obtained by using the Leibnitz rule for the kth derivative of a product.
Therefore,

$$y_k \equiv D^k y(x)\Big|_{x=0} = c_1 + c_2 k, \tag{45}$$

which is easily shown to be the general solution of equation (43).

Problem

Consider a set of k spheres so placed that each sphere intersects all the other spheres. Let c_k be the number of compartments into which space is divided. Show that

$$c_{k+1} = c_k + k^2 - k + 2.$$

Problem

Consider a collection of k boxes and k labels, with one label marked for each box. Show that the number of ways, N_k, they can be mixed such that no box has its own label is

$$N_k = (k-1)N_{k-1} + (k-1)N_{k-2}.$$

Problem

Let the single, self-interacting population model (logistic equation) be harvested; i.e., a certain constant number of the population is removed at the end of the interval $t_k = (\Delta t)k$. What is the new population equation?

Problem

A vacuum pump removes one third of the remaining air in a cylinder with each stroke. Form an equation to represent this situation. After how many strokes is just $1/1000000$ of the initial air remaining?

Problem

A population is increasing at a rate of 25 per thousand per year. Define a difference equation which describes this situation. Solve it and find the population in 20 years' time, assuming the population is now 500 million. How long will it take the population to reach 750 million?

Problem

Form and solve the difference equation defined by the sequence in which the n^{th} term is formed by adding the previous two terms and then doubling the result, and in which the first two terms are both one.

Problem

In a new colony of geese there are 10 pairs of birds, none of which produce eggs in their first year. In each subsequent year, pairs of birds which are in their second or later year have, on average, 4 eggs (2 male and 2 female). Assuming no deaths, show that the recurrence relation which describes the geese population is

$$u_{n+1} = u_n + 2u_{n-1}, \quad u_1 = 10 \text{ and } u_2 = 10,$$

where u_n represents the geese population (in pairs) at the beginning of the n^{th} year.

Problem

The growth in number of neutrons in a nuclear reaction is modelled by the recurrence relation

$$u_{n+1} = 6u_n - 8u_{n-1},$$

with initial values $u_1 = 2, u_2 = 5$, where u_n is the number at the beginning of the time interval n ($n = 1, 2, \dots$). Find the solution for u_n and hence, or otherwise, determine the value of n for which the number of neutrons reaches 10000.