

Module- III
Infinite Series

Let $\{u_n\}$ be a sequence. Then the sum of the terms $u_1 + u_2 + u_3 + \dots$ of all the terms is called an infinite series and we denote it by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$.

Remark: We cannot add all the infinite number of terms of the series in the ordinary way and has no meaning of this kind of sum.

We, thus, start by associating with the given series $\sum u_n$, a sequence $\{s_n\}$, where s_n denotes the sum of the first n terms of the series. Then,

$$s_n = u_1 + u_2 + \dots + u_n, \quad \forall n \in \mathbb{N}$$

Thus, the sequence $\{s_n\}$ is called the sequence of partial sums of the series.

The partial sums

$$\begin{aligned} s_1 &= u_1 \\ s_2 &= u_1 + u_2 \\ s_3 &= u_1 + u_2 + u_3 \\ &\vdots \quad \vdots \quad \vdots \end{aligned}$$

} may be regarded as approximation to the full infinite sum $\sum_{n=1}^{\infty} u_n$ of the series.

If the sequence $\{s_n\}$ of partial sums converges, then the series is regarded as convergent and

$\lim s_n$ is said to be the sum of the series.

If, however, $\{s_n\}$ does not tend to a limit, we say that the sum of the infinite series does not exist; the series does not converge.

Remark: An infinite series is said to converge, diverge or oscillate according as its sequence of partial sums $\{s_n\}$ converges, diverges or oscillates.

Some Examples

Ex. 1. Determine the convergence of the series

(i) $1+2+3+\dots+n+\dots$, i.e., $\sum n$.

(ii) $5-4-1+5-4-1+\dots$

(iii) Here $s_n = 1+2+3+\dots+n = \frac{n(n+1)}{2}$.

$$\lim s_n = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2} = \infty.$$

Hence the series $1+2+3+\dots$ is divergent.

(ii) Here $s_n = 5-4-1+5-4-1+5-4-1+\dots$ or terms

= 0 or 5 or 1 according ~~as~~ as the no. of terms $3m, 3m+1, 3m+2$.

Thus, in this case, s_n does not tend to a unique limit.
Hence the series is oscillatory.

Ex. 2. Geometric series:

Show that the series $1+r+r^2+r^3+\dots$

(i) converges if $|r| < 1$

(ii) diverges if $r \geq 1$; and

(iii) oscillates if $r \leq -1$.

Ans. Let $s_n = 1+r+r^2+\dots+r^n = \frac{1-r^n}{1-r}$.

When $|r| < 1$, $\lim_{n \rightarrow \infty} r^n = 0$, so that $\lim_{n \rightarrow \infty} s_n = \frac{1}{1-r}$.

Hence the series is convergent.

When $r > 1$, $\lim_{n \rightarrow \infty} r^n = \infty$. Hence

$$s_n = \frac{r^n-1}{r-1} = \frac{r^n}{r-1} - \frac{1}{r-1} \text{ so that } \lim_{n \rightarrow \infty} s_n = \infty.$$

Hence the series is divergent.

When $r = -1$, then the series becomes $1-1+1-1+1-1\dots$
which is an oscillating series.

When $r < -1$, let $r = -p$ so that $p > 1$. Then $r^n = (-1)^n p^n$

$$s_n = \frac{1-r^n}{1-r} = \frac{1-(-1)^n p^n}{1-r} \text{ and } \lim_{n \rightarrow \infty} p^n = \infty.$$

H.W Examine the following series for convergence:

1. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ (goes to ~~test~~ lead some new definition of properties soon).

2. $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \dots$

3. $6 - 10 + 4 + 6 - 10 + 4 + 6 - 10 + \dots$

4. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$

H.W A ball is dropped from a height h meters. Each time the ball hits the ground, it rebounds a distance r times the distance fallen where $0 < r < 1$. If $h=3$ meters and $r=2/3$, then find the total distance travelled by the ball.

Some general properties of Series

The truth of the following properties is self-evident and these may be regarded as axioms:

1. The convergence or divergence of an infinite series remains unaffected by the addition or removal of a finite number of its terms. This is because these terms, being the finite quantity, does not alter the nature of the infinite sum.
2. If a series, in which all the terms are positive, is convergent, then the series remains convergent even when some or all of its terms are negative. This is because the sum (infinite) ~~is~~ is clearly greatest when all the terms are positive.
3. The convergence or divergence of an infinite series remains unaffected by multiplying each term^{term} by a finite number.

Theorem 1. A necessary condition for convergence of an infinite series $\sum u_n$ is $\lim_{n \rightarrow \infty} u_n = 0$.

Ex.3. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots$

$$u_n = \frac{n}{n+1} \Rightarrow \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0.$$

By Theorem 1, the series $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots$ is not convergent.

Cauchy's General Principle of Convergence for Series

A necessary and sufficient condition for the convergence of an infinite series $\sum_{n=1}^{\infty} u_n$ is that the sequence of its partial sums $\{s_n\}$ is convergent.

Remark The convergence of an infinite series may be derived by using the knowledge of sequences.

Theorem 2.

A series $\sum u_n$ converges iff to each $\epsilon > 0$, there exists a positive integer m such that

$$|u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \epsilon \quad \forall n \geq m \text{ and } p \geq 1.$$

Note By a Cauchy's general principle of convergence (for sequences) the sequence $\{s_n\}$ of partial sums of $\sum u_n$ converges iff to each $\epsilon > 0$, there exists a positive integer m , such that

$$|s_{n+p} - s_n| < \epsilon \quad \forall n \geq m \text{ and } p \geq 1,$$

that is,

$$|u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \epsilon \quad \forall n \geq m \text{ and } p \geq 1.$$

Example 4. Show that the series $\sum v_n$ does not converge.

Pf/: If possible, let the series converge. So for any given $\epsilon > 0$, say $\epsilon = v_4$, there exists a positive integer m , such that

$$\left| \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p} \right| < \epsilon = v_4 \quad \forall n \geq m \text{ and } p \geq 1.$$

In particular, if $n = m$ and $p = m$, we get

$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p} = \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{m+m} \geq m \cdot \frac{1}{2m} = \frac{1}{2} > \epsilon,$$

a contradiction. Hence the given series does not converge.

Ex If $u_n > 0$ and $\sum u_n$ is cpt w/w the series S, then prove that

H-W . $\sum \frac{u_n}{u_1 + u_2 + \dots + u_n} \Leftrightarrow \text{S}$ is convergent.

Positive Term Series

Why we prefer positive term series? One of the reason is the sequence of partial sums is monotonic increasing. One of the simple sequence we deal in many mathematical models.

Let $\sum u_n$ be an infinite series of positive terms and $\{s_n\}$ be a sequence of partial sums, so that

$$s_n = u_1 + u_2 + \dots + u_n \geq 0 \forall n.$$

$$\Rightarrow s_n - s_{n-1} = u_n \geq 0$$

$\Rightarrow s_n \geq s_{n-1} \Rightarrow$ The sequence $\{s_n\}$ of partial sums of a positive terms series is a monotonic increasing sequence.

Note: Since a monotonic increasing sequence can converge, diverge but cannot oscillate, we shall be dealing with two cases: convergence and divergence of positive term series.

Theorem: A positive term series converges iff the sequence of its partial sums is bounded above.

Remark: We can deal w/w negative term series but bounded below property is required.

Theorem: If a series $\sum u_n$ of positive monotonic decreasing terms converges, then $u_n \rightarrow 0$ and $n u_n \rightarrow 0$ as $n \rightarrow \infty$.

There is an important theorem, researchers use this theorem to study the asymptotic behaviour of solutions of difference equations.

In this theorem, the terms are decreasing but the sequence of partial sums is increasing.

Try to recall sequence theorem.

clearly sequence of sums is monotonically increasing.

Geometric Series

The positive term geometric series $1+r+r^2+\dots$ converges for $r < 1$, and diverges to $+\infty$ for $r \geq 1$.

We have seen the proof in page no. 2 i.e. Ex-2

Theorem (p-series Test)

A positive term series $\sum \frac{1}{n^p}$ is convergent if and only if $p > 1$. (Proof - not required)

→ p-series test helps us to derive ~~many~~ tests for convergence and divergence of many infinite series by comparing.

We see them now.

Comparison Test (first type).

Result 1. If $\sum u_n$ and $\sum v_n$ are two positive term series, and $K \neq 0$, a fixed positive number (independent of n) and there exists a positive integer m such that

$$u_n \leq K v_n, \quad \forall n \geq m.$$

Then the following holds :

- (i) if $\sum v_n$ is convergent, then $\sum u_n$ is convergent.
- (ii) if $\sum u_n$ is divergent, then $\sum v_n$ is divergent.

Result 2. (Limit form) If $\sum u_n$ and $\sum v_n$ are two positive term series such that

$$\lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n} \right) = l,$$

where l is a nonzero finite number, then two series converge or diverge together.

Comparison Test (second type).

Result:3 If $\sum u_n$ and $\sum v_n$ are two positive term series and there exists a positive integer m such that

$$\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}} \quad \forall n \geq m,$$

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(i) $\sum v_n$ convergent $\Rightarrow \sum u_n$ convergent.

(ii) $\sum u_n$ divergent $\Rightarrow \sum v_n$ divergent.

Some Examples

Ex-5. Show that the series $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$ is convergent.

Pf: We have $\frac{1}{2!} = \frac{1}{2}$, $\frac{1}{3!} < \frac{1}{2^2}$, $\frac{1}{4!} < \frac{1}{2^3}$, ..., $\frac{1}{n!} < \frac{1}{2^{n-1}}$.

$$\Rightarrow 1 + \frac{1}{2!} + \frac{1}{3!} + \dots < 1 + \underbrace{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots}_{\text{geometric series with } r = \frac{1}{2} \text{ & } 0 < r < 1.}$$

by Test/Result-1,
this series convergent. \Leftarrow Hence convergent.

Ex-6. Show that the series

$$\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \dots + \frac{1}{(\log n)^p} + \dots$$

is divergent for $p > 0$.

Pf: Since $\lim_{n \rightarrow \infty} \frac{(\log n)^p}{n} = 0$, then $(\log n)^p < n \quad \forall n > 1$.

\Rightarrow So, $\frac{1}{(\log n)^p} > \frac{1}{n}$. Consequently,

$$\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \dots + \frac{1}{(\log n)^p} > \underbrace{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots}_{\text{divergent series.}}$$

by
Result-1,
 n divergent.

Ex. 7. Investigate the behaviour of the series $\sin Y_n$.

Let $u_n = \sin Y_n$ and $v_n = Y_n$. (we have set them).

Now, $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{\sin Y_n}{Y_n} = 1$.

Then, by Result 2, both the series behave alike. Since $\sum Y_n$ is a divergent series, then the series $\sum \sin Y_n$ is divergent.

Ex. 8. Test the convergence of the series whose n -th term is $[(n^3+1)^{\frac{1}{3}} - n]$.

$$\begin{aligned} \text{Set } u_n &= (n^3+1)^{\frac{1}{3}} - n = n \left[\left(1 + \frac{1}{n^3}\right)^{\frac{1}{3}} - 1 \right] \\ &= n \left\{ \frac{1}{3n^3} + \dots \right\} \quad (\text{we shall get higher powers of } n \text{ in denominator}) \\ &= \frac{1}{3n^2} + \dots \end{aligned}$$

Then set $v_n = Y_{n^2}$. Then $\sum v_n$ is cgt (keep in mind)

Now, $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{3}$.

Hence by Test/Result 2, both the series converge or diverge together. Since $\sum v_n$ is cgt, then $\sum u_n$ is cgt.

Ex. 9. Test the cgt. of the series $\sum \frac{1}{n^{1/Y_n}}$.

Set $u_n = \frac{1}{n^{1/Y_n}}$ and $v_n = Y_n$.

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{n^{Y_n}} = 1.$$

Hence, both the series have same behaviour.

Since $\sum v_n$ is dgt., then $\sum u_n$ is dgt.

(H.W) (i) $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \dots$ cgt

(ii) $\frac{1}{\sqrt{1 \cdot 2}} + \frac{1}{\sqrt{2 \cdot 3}} + \frac{1}{\sqrt{3 \cdot 4}} + \dots$ dgt.

(iii) $\frac{1}{\sqrt{4 \cdot 6}} + \frac{1}{\sqrt{6 \cdot 8}} + \frac{1}{\sqrt{8 \cdot 10}} + \frac{1}{\sqrt{10 \cdot 12}} + \dots$ cgt.

iv) $\sum \frac{n+1}{n^p}$ cgt. for $p > 2$.

v) $\sum (\sqrt{n^4+1} - \sqrt{n^4-1})$ cgt.

vi) $\sum \sin y_n$ cgt.

vii) $\sum \cos y_n$ dgt.

viii) $\sum \frac{1}{\sqrt{n}} \tan y_n$ cgt.

ix) $\sum_{n=1}^{\infty} e^{-n^2}$ cgt.

x) $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$ cgt.

xi) $\sum \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$ cgt.

H.W. Show that the series $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \dots + \frac{1}{n^n} + \dots$ is cgt.

Cauchy's Root Test

If $\sum u_n$ is a positive term series such that

$$\lim_{n \rightarrow \infty} u_n^{1/n} = l,$$

then the series

(i) converges, if $l < 1$

(ii) diverges, if $l > 1$, and

(iii) the test fails to give any definite information, if $l = 1$.

Ex 10 $\sum \left(1 + \frac{1}{n}\right)^{-n^{1/2}}$.

$$u_n = \frac{1}{\left(1 + \frac{1}{n}\right)^{n^{1/2}}} \Rightarrow \lim_{n \rightarrow \infty} (u_n)^{1/n} = \frac{1}{\left(1 + \frac{1}{n}\right)^{1/\sqrt{n}}} = \frac{1}{e} < 1.$$

Hence the series cgt.

D'Alembert's Ratio test

If $\sum u_n$ is a positive term series such that

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = l,$$

then the series

- (i) converges if $l < 1$
- (ii) diverges if $l > 1$, and
- (iii) the test fails if $l = 1$.

Ex-11 $\sum \underbrace{\frac{n^2-1}{n^2+1} \cdot x^n}_{u_n}, x > 0.$

$$\therefore u_n \Rightarrow \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x.$$

Hence by D'Alembert's ratio test, the sum is cst if $x < 1$,
dgt. if $x \geq 1$ and the test fails if $x = 1$.

So, we consider the particular case $x = 1$. When $x = 1$,

$$u_n = \frac{n^2-1}{n^2+1}, \text{ and } \lim_{n \rightarrow \infty} u_n = 1 \neq 0.$$

Hence the sum is divergent.

∴ Then, the sum $\sum \frac{n^2-1}{n^2+1} x^n$ is cst for $x < 1$ and dgt for $x \geq 1$.

Remark - Cauchy's root test is stronger than D'Alembert's ratio test and may succeed when ratio test fails.

Ex. to the above Remark : $\sum u_n$, where $u_{2n-1} = \frac{1}{2^{2n-1}}, u_{2n} = \frac{1}{2^n} k_n$

Raabe's Test :

If $\sum u_n$ is a positive term series such that

$$\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = l,$$

then the series

- (i) converges if $l > 1$
- (ii) diverges if $l < 1$, and
- (iii) the test fails if $l = 1$.

Ex. 12 Test the convergence or divergence of the series

$$\frac{\alpha}{\beta} + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \dots$$

$$u_n = \frac{(1+\alpha)(2+\alpha)(3+\alpha) \dots (n-1+\alpha)}{(1+\beta)(2+\beta)(3+\beta) \dots (n-1+\beta)}.$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{n+\alpha}{n+\beta} = 1.$$

Hence the ratio test fails.

Again

$$\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = \beta - \alpha.$$

Then, by Raabe's test, the series converges if $\beta - \alpha > 1$, diverges if $\beta - \alpha < 1$. The test fails for $\beta - \alpha = 1$.

However, for $\beta - \alpha = 1$, i.e. $\beta = \alpha + 1$, the series becomes

$$\frac{\alpha}{\alpha+1} + \frac{1+\alpha}{2+\alpha} + \frac{1+\alpha}{3+\alpha} + \dots = \sum \frac{1+\alpha}{n+\alpha},$$

which is divergent, by comparison with $\sum 1/n$.

Logarithmic Test.

If $\sum u_n$ is a positive term series such that

$$\lim_{n \rightarrow \infty} \left(n \log \frac{u_n}{u_{n+1}} \right) = l,$$

then the positive term series cgs. for $l > 1$, and dgs. for $l < 1$.

Ex. 13 Test the cgs. of the series

$$1 + \frac{x}{1!} + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \dots \text{ for } x > 0.$$

Ignoring the first term, set $u_n = \frac{n^n x^n}{n!}$. Then $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = ex$.

By ratio test, the series converges for $ex < 1$ or $x < 1/e$, and

diverges for $x > 1/e$. For $x = 1/e$, $\frac{u_n}{u_{n+1}} = \left(\frac{n}{n+1}\right)^n e$. Now,

$\lim_{n \rightarrow \infty} n \log \frac{u_n}{u_{n+1}} = \frac{1}{2} < 1$. Thus, by Logarithmic test, the series diverges for $x = 1/e$. Hence, the series cgs. for $x < 1/e$ and diverges for $x \geq 1/e$.

Integral Test.

$\int_{\alpha}^{\infty} u(x) dx$ cgs. if $v(x) = \int_{\alpha}^x u(t) dt$ converges or tends to a finite limit as $x \rightarrow \infty$.

Let $u(x) > 0$ for all $x \geq \alpha$. Then

$$v(t) = \int_{\alpha}^t u(x) dx$$

is increasing function of t .

If we set $v(t)$ as v_t , then $\{v_t\}$ is an increasing / nondecreasing sequence which is convergent only when it is bounded above.

That is,

$\int_{\alpha}^{\infty} u(x) dx$ converges iff it is bounded above.

That is, \exists a positive number K such that

$$\int_{\alpha}^t u(x) dx \leq K \quad \forall t \geq \alpha.$$

Cauchy's Integral Test

If u is a nonnegative monotonic decreasing integrable function such that $u(n) = u_n$ for all positive integral values of n , then the series $\sum u_n$ and $\int_{\alpha}^{\infty} u(x) dx$ converge or diverge together.

Example 14. Show that the series $\sum \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Ans Let $u(x) = \frac{1}{x^p}$, so that for $x \geq 1$, the function u is a nonnegative monotonic decreasing function such that

$$u_n = u(n) = \frac{1}{n^p}, n \in \mathbb{N}.$$

By integral test, $\sum_{n=1}^{\infty} u_n$ and $\int_1^{\infty} u(t) dt$ converge or diverge together.

We are asked to test the cgs. of $\sum_{n=1}^{\infty} u_n$, $u_n = \frac{1}{n^p}$.

So we test the cgs. of the infinite integral.

$$\int_1^x u(t) dt = \int_1^x \frac{1}{t^p} dt = \begin{cases} \frac{1}{1-p} (x^{1-p} - 1) & \text{if } p \neq 1 \\ \log x & \text{if } p = 1. \end{cases}$$

$$\Rightarrow \int_1^{\infty} u(t) dt = \lim_{x \rightarrow \infty} \int_1^x \frac{1}{t^p} dt = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } 0 < p \leq 1. \end{cases}$$

Then, $\int_1^{\infty} u(t) dt$ converges if $p > 1$ and dgs. if $p \leq 1$.

consequently, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is cgt for $p > 1$ and dgt. for $p \leq 1$.

Gauss's Test

If $\sum u_n$ is a positive term series with

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{r_n}{n^p}, \quad \alpha > 0, \beta > 0, \{r_n\} \text{ is a bdd. sequence}$$

then

- (i) for $\alpha \neq 1$, $\sum u_n$ cgs. if $\alpha > 1$, dgs if $\alpha < 1$ whatever β may be.
- (ii) for $\alpha = 1$, $\sum u_n$ cgs if $\beta > 1$ and dgs if $\beta \leq 1$.

Remark (i) Gauss's test is very useful and may be used after the failure of Raabe's test or may be used directly without using other tests.

(ii) If $\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{r_n}{n^2} + \frac{s_n}{n^3} + \dots$, where α, β, r, \dots

are independent of n , then we can write

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{r_n}{n^2} \text{ where } r_n = r + \delta/n + \dots$$

so that $r_n = r$, i.e., $\{r_n\}$ is a bounded sequence.

Thus, for Gauss's test, one may expand u_n/u_{n+1} in the powers of n as in (*).

Ex 15. Test the cgt. of the series

$$\sum \frac{1^2 3^2 \cdots (2n-1)^2}{2^2 4^2 \cdots (2n)^2} x^{n-1}, x > 0.$$

Here $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{2}$. Hence by Ratio test, the series cgs if $x < 1$ and dgs if $x > 1$. Now, for $x=1$,

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}$$

$$\therefore \lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = \lim_{n \rightarrow \infty} \frac{\frac{u_n}{u_{n+1}} - 1}{\frac{1}{n}} \cdot n = \lim_{n \rightarrow \infty} \frac{u_n^2 + 3n}{(2n+1)^2} = 1.$$

Hence Raabe's test fails. Let us, now, apply Gauss's test.

$$\begin{aligned} \frac{u_n}{u_{n+1}} &= \left(1 + \frac{1}{n} \right)^2 \left(1 + \frac{1}{2n} \right)^{-2} \\ &= 1 + \frac{1}{n} - \frac{1}{4n^2} + \dots \text{ higher powers of } \frac{1}{n}. \end{aligned}$$

Hence the series diverges (by Gauss's test).

Then, the ~~series~~ given series is cgt. for $x < 1$ and dgt. for $x \geq 1$.

H.W Test the cgt. of the hypergeometric series:

$$1 + \frac{\alpha \cdot \beta}{1 \cdot r} x + \frac{\alpha(\alpha+1) \beta(\beta+1)}{1 \cdot 2 \cdot r(r+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2) \beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot r(r+1)(r+2)} x^3 + \dots$$

for all positive values of x ; α, β, r being all positive.

Ans. for $x < 1$, the series cgs.

for $x > 1$, the series dgs.

for $x=1$, the series cgs if $r > (\alpha+\beta)$
and dgs if $r \leq (\alpha+\beta)$.

Alternating Series

A series whose terms are alternatively positive and negative is called a alternating series.

For example, $1 - 2 + 3 - 4 + 5 - 6 + \dots$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

$$1 - x + x^2 - x^3 + x^4 - \dots$$

Leibnitz Test

of the alternating series

$$u_1 - u_2 + u_3 - u_4 + \dots, (u_n > 0 \forall n)$$

is such that

$$(i) u_{n+1} \leq u_n \quad \forall n, \quad (ii) \lim_{n \rightarrow \infty} u_n = 0,$$

then the series is convergent.

Example 16 - Show that the series

$$\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots \quad (:= \sum \frac{(-1)^{n-1}}{n^p})$$

converges for $p > 0$.

$$\text{Let } u_n = (-1)^{n-1} \cdot \frac{1}{n^p}. \quad \text{Then } u_{n+1} \leq u_n \quad \forall n.$$

$$\text{and } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n^p} = 0 \quad (\because p > 0).$$

Hence by Leibnitz test, the alternating series e.g.

Absolute convergence

A series $\sum u_n$ is said to be absolutely convergent if the series $\sum |u_n|$ is convergent.

conditional convergence

A series which is convergent but not absolutely convergent is called a conditionally convergent series.

Some Examples:

Ex. 17. The series

$$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \dots$$

is absolutely convergent because the series

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3}$$

is convergent.

Ex-18 The series

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \dots$$

is convergent by Leibnitz test, but the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \dots$$

is divergent. Thus, the given series is conditionally convergent.

Theorem : Every absolutely cgt. series is cgt.

Remark The divergence of $\sum |u_n|$ need not imply that $\sum u_n$ is divergent.

Ex-19. $\sum \frac{(-1)^{n+1}}{n}$. \rightarrow cgt - (by Leibnitz test)
 However, if we consider the absolute terms, then
 the sum $\sum \frac{1}{n}$ is divergent.

Ex-20. Show that for any fixed value of x , the

series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is cgt.

Let $u_n = \frac{\sin nx}{n^2}$ so that $|u_n| = \frac{|\sin nx|}{n^2}$.

Now, $\left| \frac{\sin nx}{n^2} \right| \leq \frac{1}{n^2}$ & $\sum \frac{1}{n^2}$ is cgt.

Hence by comparison test, $\left| \frac{\sin nx}{n^2} \right|$ is cgt.

Therefore, Consequently, the above theorem implies that

the series $\sum \frac{\sin nx}{n^2}$ is convergent.

H.W. Show that the series

$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

converges absolutely for all values of x .