

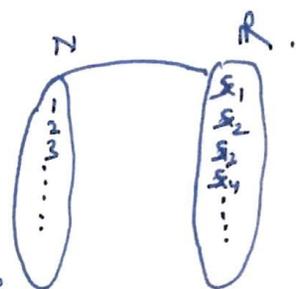
Module II -1-

Sequences (Real)

Definition A function whose domain is the set N of natural numbers and range a set of real numbers is called a real sequence.

Symbolically, $S: N \rightarrow \mathbb{R}$.

Since the domain for a sequence is always N , we specify a sequence by the values $S_n, n \in N$. Then, we can write a sequence in the form $\{S_n\}_{n \in N}$.



or $\{S_n\}_{n=1}^{\infty}$ or $\{S_1, S_2, S_3, \dots, S_n, \dots\}$
1st element, 2nd element, 3rd element, ..., n-th element.

→ We treat m-th term & n-th terms S_m & S_n as distinct terms even as $S_m = S_n$.

→ The terms of a sequence are arranged in a definite order as 1st, 2nd, 3rd, ..., n-th, ... & the terms occurring different positions are treated as distinct terms even they have the same values.

→ The number of terms in a sequence is always infinite.

→ In other words, we define a sequence as an ordered set of real numbers whose members can be put in a one-one correspondence with the set of natural numbers.

→ However, a sequence may have only a finite number of distinct terms.

Some Examples

1. $S_n = (-1)^n$ $\xrightarrow{\text{sequence}}$ $\{(-1)^n\}_{n \in N}$. only two elements 1, -1, both distinct.

2. $S_n = \frac{1}{n}$ $\xrightarrow{\text{consider the sequence}}$ $\{\frac{1}{n}\}_{n \in N}$. infinite no. of elements, all are distinct.

3. $S_n = (1 + \frac{1}{n})^n$, 4. $S_n = 1 + (-1)^n$ 5. $S_n = 1$ 6. $S_n = \frac{(-1)^{n+1}}{n!}$
 $n \in N$ $n \in N$ $n \in N$ $n \in N$

Range of a sequence:

The range of a sequence is the set consisting of all distinct elements of a sequence, without repetition or without regard to the position of a term.

Thus, the set (range) may be a finite or an infinite set.

Bounds of a sequence:

A sequence $\{s_n\}$ is said to be bounded above if there exists a real number k such that $s_n \leq k \forall n \in \mathbb{N}$.

A sequence $\{s_n\}$ is said to be bounded below if there exists a real number k such that $s_n \geq k \forall n \in \mathbb{N}$.

A sequence is said to be bounded if it is both bounded above and bounded below. So, obviously, we see that the ranges are bounded.

Convergence of Sequences

Def: A sequence $\{s_n\}$ is said to converge to a real number l (sometimes, we may l to be a limit of the sequence) if for each $\epsilon > 0$, there exists a positive integer m (depending on ϵ) such that

$$|s_n - l| < \epsilon \text{ for all } n \geq m.$$

We denote the convergence/limit of the sequence as

$$s_n \rightarrow l \text{ as } n \rightarrow \infty \text{ or } \lim_{n \rightarrow \infty} s_n = l.$$

what is the meaning of the above definition: ???

→ From some stage onwards, the differences between s_n and l can be made less than ~~by~~ any preassigned number ϵ , however small.

That is, given any positive real ϵ , no matter how small it is, there exists a +ve integer m (finite value) such that the terms after m / m -th term onwards, that is, $s_m, s_{m+1}, s_{m+2}, \dots$ remains arbitrary close to l .

We say l to be the limit point of the sequence.

→ For any $\epsilon > 0$, at the most a finite number of terms (depending on the choice of ϵ) of the sequence can lie outside $(l-\epsilon, l+\epsilon)$, that is, there is at the most a finite number of n 's for which

$$s_n \leq l-\epsilon \text{ and } s_n \geq l+\epsilon.$$

→ Since $l-\epsilon < s_n < l+\epsilon$ for all $n \geq m$, then we have the property that $s_n < l+\epsilon$ for infinite number of terms, we have the following observation:

infinite number of terms $\in (l-\epsilon, l+\epsilon)$.

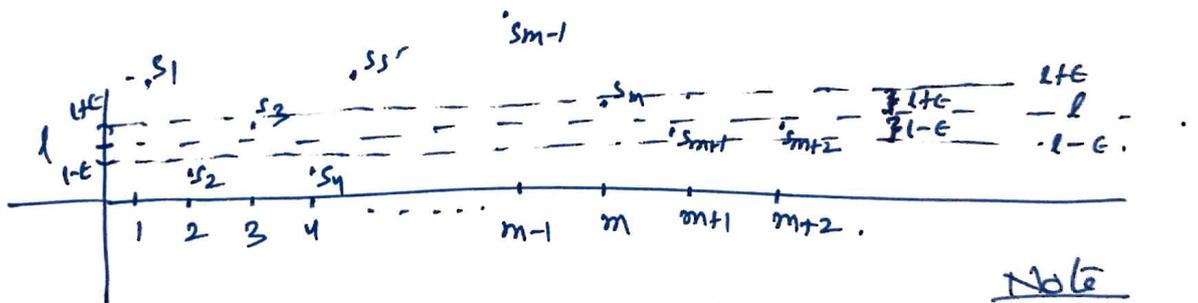
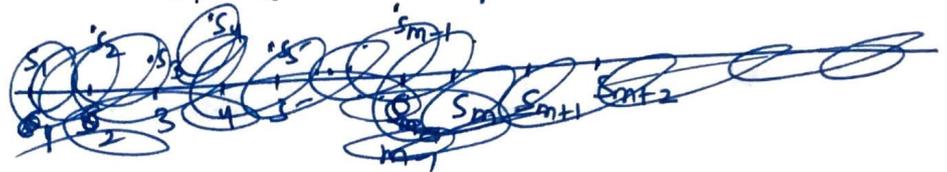
Theorem 1. Every convergent sequence is bounded.

Proof. Let a sequence $\{s_n\}$ converge to the limit l .

Let $\epsilon > 0$ be a given number for which there exists a positive integer m such that

$$|s_n - l| < \epsilon \text{ for } n \geq m.$$

or $l - \epsilon < s_n < l + \epsilon \text{ for } n \geq m. \dots (1)$



Let $g = \min \{ l - \epsilon, s_1, s_2, \dots, s_{m-1} \}$

Note
 $\Rightarrow g \leq l - \epsilon$

and $G = \max \{ l + \epsilon, s_1, s_2, \dots, s_{m-1} \}$.

$\Rightarrow G \geq l + \epsilon$.

Then we have (follows from (1)),

$$g \leq s_n \leq G \text{ for } n \geq 1.$$

Hence G is a bounded sequence. The theorem is proved.

Remark The converse of Theorem 1 may not be

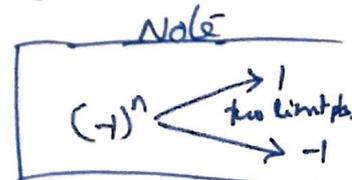
true.

For example, consider the sequence

$$s_n = (-1)^n, n \in \mathbb{N}.$$

This sequence is bounded but not convergent.

If possible, let $s_n \rightarrow l$. Then for $\epsilon = 1$, $\exists m \in \mathbb{N}$ such that $|s_n - l| < 1 \forall n \geq m$, that is, $|(-1)^n - l| < 1 \forall n \geq m$, that is, in particular for $n = 2m$, we have $|(-1)^{2m} - l| < 1$ and for $n = 2m+1$, we have $|(-1)^{2m+1} - l| < 1$. Consequently,



we have

$$|1-l| < 1 \text{ and } |1+l| < 1.$$

On the other hand, by triangular inequality, we have

$$\begin{aligned} 2 = 1-l+1+l &< |1-l+1+l| \\ &\leq |1-l| + |1+l| < 1+1 = 2, \end{aligned}$$

a contradiction.

Hence the converse of Theorem 1 is not necessarily true.

Theorem 2. A sequence cannot converge to more than one limit.

Proof. If possible, ~~the~~ let there exists a sequence $\{s_n\}$ converges to two \neq different limits l and l' .

$$\text{Set } \epsilon = \frac{1}{2} |l-l'| > 0.$$

Since the sequence $\{s_n\}$ converges to l and l' , there exist positive integers (natural numbers) m and m' such that

$$\begin{aligned} |s_n - l| &< \epsilon \text{ for } n \geq m \\ \text{and } |s_n - l'| &< \epsilon \text{ for } n \geq m'. \end{aligned}$$

~~Set~~ Hence for $n \geq m_1 = \max\{m, m'\}$, we have

$$|s_n - l| < \epsilon \text{ and } |s_n - l'| < \epsilon \text{ for } n \geq m_1.$$

consequently, we have

$$\begin{aligned} |l-l'| &= |l-s_n+s_n-l'| \leq |l-s_n| + |s_n-l'| \\ &= |s_n-l| + |s_n-l'| \\ &< \epsilon + \epsilon = 2\epsilon = |l-l'|, \end{aligned}$$

a contradiction.

Thus, the sequence $\{s_n\}$ cannot converge to two limits. The theorem is proved.

Remark: We have observed that a sequence converges to a number which is a limit point of the sequence. This limit point is unique. We say, sometimes, as the limit of the sequence, and symbolically, we write/express as

$$\lim_{n \rightarrow \infty} S_n = l \quad \text{or} \quad S_n \rightarrow l \text{ as } n \rightarrow \infty$$

or $\lim S_n = l.$

The above two theorems give the following beautiful criteria.

Theorem 3. Every convergent sequence is bounded and has a limit point.

Limit Points of a Sequence

A real number ξ is said to be a limit point of a sequence $\{S_n\}$ if every nbhd. of ξ contains an infinite number of members of the sequence.

Thus, ξ is a limit point of the sequence if given any positive number ϵ , however small,

$$S_n \in (\xi - \epsilon, \xi + \epsilon)$$

for an infinite values of n , i.e.,

$$|S_n - \xi| < \epsilon \quad \text{for infinite no. of values of } n.$$

In other words, S_n is arbitrary close to ξ for an infinite number of values of n or infinite no. of members of the sequence are very close to ξ .

Note: A number ξ is not a limit point of the sequence $\{S_n\}$ if there exists a number ~~ϵ~~ $\epsilon > 0$ such that $S_n \in (\xi - \epsilon, \xi + \epsilon)$ for at most a finite no. of values of n .

Some Examples

(1) $\{s_n\}$, $s_n = 1 \forall n$. Constant sequence.

This has only ~~one~~ one limit point 1.
However, the range set $\{1\}$ which has no limit point.

(2) $\{s_n\}$, $s_n = \frac{1}{n}$, $n \in \mathbb{N}$. 0 is the limit point,
which is also a limit point of the range
 $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$.

(3). $\{s_n\}$, $s_n = 1 + (-1)^n$, $n \in \mathbb{N}$. 0 and 2 are the limit
points. However, the range set $\{0, 2\}$ has
no limit points.

(4) $\{s_n\}$, $s_n = (-1)^n$, $n \in \mathbb{N}$. 1, -1 are the limit points.
However, the range set $\{-1, 1\}$ has no limit points.

(5-). $\{s_n\} = (-1)^n \left(1 + \frac{1}{n}\right)$, $n \in \mathbb{N}$. 1, -1 are the limit points
which are the limit points of ~~the~~ its range set.

Existence of Limit Points

We note that members of a sequence forms a set (the range set / subset of the range set). Thus, all theorems relating to bounded and limit points also works ~~of~~ for sequences, with suitable modifications.

Bolzano-Weierstrass Theorem :

Every bounded sequence has a limit point.

Proof. Let $\{s_n\}$ be a bounded sequence and

$$S = \{s_n; n \in \mathbb{N}\}$$

be its range. Since the sequence is bounded, then its range set is also bounded

Now, we have two possibilities :

(i) S is finite

(ii) S is infinite

We deal with ~~all~~ both the possibilities given

above :

(i) If S is finite, then there exists at least one ~~more~~ member $\xi \in S$ such that $s_n = \xi$ for an infinite number of values of n . This means that every nbhd $(\xi - \epsilon, \xi + \epsilon)$ of ξ contains $s_n (= \xi)$ for an infinite number of values of n .

Thus, ξ is the limit point of $\{s_n\}$.

(ii) When S is infinite, since it is bounded, then by the Bolzano-Weierstrass Theorem for sets, it has a limit point; say ξ .

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Again, since ξ is a limit point of S , then every nbhd. $(\xi - \epsilon, \xi + \epsilon)$ of ξ contains an infinite member of S , that is, $s_n \in (\xi - \epsilon, \xi + \epsilon)$ for infinite values of n . Thus, ξ is a limit point of the sequence

Remark The converse of the above theorem is not necessarily true. For example, the sequence

$\{1, 2, 1, 4, 1, 6, 1, 8, \dots\}$

has a unique limit point 1 but not bounded above.

Theorem: The set of limit points of a bounded ^{seq.} sequence has the greatest and the least members.

Remark: The greatest and the smallest of the limit points of a bounded sequence are respectively, called the upper limit and the lower limit.

Illustrations

(i) $\{s_n\}$, $s_n = (-1)^n, n \in \mathbb{N}$. bounded, $-1 \leq s_n \leq 1 \forall n \in \mathbb{N}$.
 -1 and 1 are the limit points.
 upper limit is 1 and lower limit is -1 .

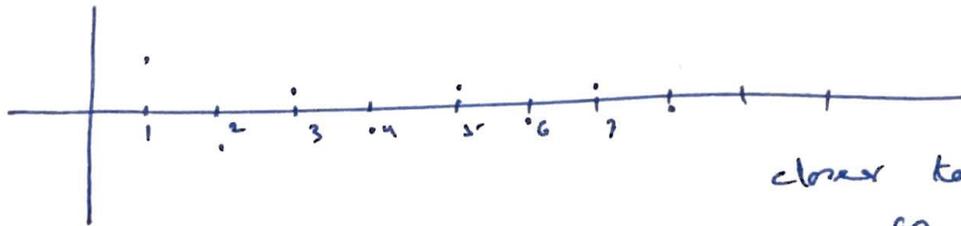
(ii) $\{s_n\}$, $s_n = 1 + (-1)^n, n \in \mathbb{N}$. bounded, $0 \leq s_n \leq 2, \forall n \in \mathbb{N}$.
 0 and 2 are the limit points.
 Upper limit = 2 , lower limit = 0 .

(iii) $\{s_n\}$, $s_n = \frac{(-1)^{n+1}}{n}, n \in \mathbb{N}$. $\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots\}$ bounded.

$$-\frac{1}{2} \leq s_n \leq 1 \forall n \in \mathbb{N}.$$

Here 0 is the limit point, upper limit & lower limit coincide with 0 , and so

$$\lim_{n \rightarrow \infty} s_n = 0.$$



closer to zero
as n increases
or as we go right.

(iv) $\{S_n\}$, $S_n = n^2$, $n \in \mathbb{N}$. $\{1, 4, 9, 16, 25, \dots\}$

lower bound is 1, upper bound? so not bounded above. So the sequence $\{S_n\}$ has no limit point.

Limit Inferior and Limit Superior of Sequences:

We have observed from the definition of limits of a sequence, that the limiting behaviour of any sequence $\{a_n\}$ of real numbers, depends only ~~on~~ on sets of the form $\{a_n; n > m\}$, that is, $\{a_m, a_{m+1}, a_{m+2}, \dots\}$ on view of this, we give the following definition:

Definition:

Let $\{a_n\}$ be a sequence of real numbers (not necessarily bounded). We define

$$\liminf_{n \rightarrow \infty} a_n = \sup_n \inf \{a_n, a_{n+1}, a_{n+2}, \dots\}$$

$$\limsup_{n \rightarrow \infty} a_n = \inf_n \sup \{a_n, a_{n+1}, a_{n+2}, \dots\}$$

as the limit inferior and limit superior respectively of the sequence $\{a_n\}$

we denote:

limit inferior $\lim_{n \rightarrow \infty} a_n$ or $\underline{\lim} a_n$

limit superior $\overline{\lim}_{n \rightarrow \infty} a_n$ or $\overline{\lim} a_n$

If we use the following notations for the sequence $\{a_n\}$,
for each $n \in \mathbb{N}$,

$$A_n = \inf \{a_n, a_{n+1}, a_{n+2}, \dots\}.$$

$$\bar{A}_n = \sup \{a_n, a_{n+1}, a_{n+2}, \dots\},$$

then we have

$$\underline{\lim} a_n = \sup_n A_n$$

$$\bar{\lim} a_n = \inf_n \bar{A}_n.$$

Remark.

Since $\{a_{n+1}, a_{n+2}, \dots\} \subseteq \{a_n, a_{n+1}, a_{n+2}, \dots\}$,
then taking limit sup. & limit inf both sides, we get

$$\underline{A}_{n+1} \geq \underline{A}_n \quad \wedge \quad \bar{A}_{n+1} \leq \bar{A}_n.$$

$$\underline{A}_1 \leq \underline{A}_2 \leq \underline{A}_3 \leq \dots \leq \underline{A}_n \leq \underline{A}_{n+1} \leq \dots$$

sequence $\{\underline{A}_n\}$ monotonically
increasing.

$$\dots \leq \bar{A}_{n+1} \leq \bar{A}_n \leq \bar{A}_{n-1} \leq \dots \leq \bar{A}_2 \leq \bar{A}_1$$

sequence $\{\bar{A}_n\}$ is
monotonically decreasing.

Theorem: If $\{a_n\}$ is any sequence, then
 $\inf a_n \leq \underline{\lim} a_n \leq \bar{\lim} a_n \leq \sup a_n.$

Corollary: If a sequence $\{a_n\}$ is bounded, then there's
limit inferior & limit superior are both finite,
that is,
 $-\infty < \underline{\lim} a_n \leq \bar{\lim} a_n < \infty.$

Theorem If $\{a_n\}$ is any sequence, then

$$\underline{\lim} (-a_n) = -\bar{\lim} a_n$$

and $\bar{\lim} (-a_n) = -\underline{\lim} a_n$.

Illustrations.

(i) $a_n = (-1)^n, n \in \mathbb{N}$. $\{(-1)^n\}$. Then

$$\underline{A}_n = -1 \text{ and } \bar{A}_n = 1, \forall n \in \mathbb{N}.$$

$$\underline{\lim} a_n = \sup \underline{A}_n = -1, \quad \bar{\lim} a_n = \inf \bar{A}_n = 1.$$

(ii) $\{1+(-1)^n\}_{n=1}^{\infty}$ $a_n = 1+(-1)^n, \forall n \in \mathbb{N}$. Then

$$\underline{A}_n = \inf \{1+(-1)^n, 1+(-1)^{n+1}, 1+(-1)^{n+2}, \dots\} = 0.$$

$$\bar{A}_n = \sup \{1+(-1)^n, 1+(-1)^{n+1}, 1+(-1)^{n+2}, \dots\} = 2$$

for each $n \in \mathbb{N}$. Hence

$$\underline{\lim} a_n = 0 \text{ and } \bar{\lim} a_n = 2.$$

Homework (To do yourself. Try to do yourself. If fails, then have discussion among your friends. If fails, again, I am here to solve your problems/doubts.)

(i) $\{n\}_{n=1}^{\infty}$ (ii) $\{(-1)^n\}_{n \in \mathbb{N}}$ (iii) $\left\{\frac{(-1)^n}{n^2}\right\}_{n \in \mathbb{N}}$.

$$\underline{\lim} a_n = \infty$$

$$\underline{\lim} a_n = -\infty$$

$$\underline{\lim} a_n = 0.$$

$$\bar{\lim} a_n = \infty.$$

$$\bar{\lim} a_n = \infty.$$

$$\bar{\lim} a_n = 0.$$

(iv) $\{(-1)^n (1 + \frac{1}{n})\}_{n \in \mathbb{N}}$.

(v) $\{n(1+(-1)^n)\}_{n \in \mathbb{N}}$

(vi) $\left\{\sin \frac{n\pi}{3}\right\}_{n \in \mathbb{N}}$.

$$\underline{\lim} a_n = -1$$

$$\underline{\lim} a_n = 0$$

$$\underline{\lim} a_n = -\frac{\sqrt{3}}{2}$$

$$\bar{\lim} a_n = 1$$

$$\bar{\lim} a_n = \infty$$

$$\bar{\lim} a_n = \frac{\sqrt{3}}{2}$$

Example: $\left\{ \sin \frac{n\pi}{2} + \frac{(-1)^n}{n} \right\}_{n \in \mathbb{N}}$.

Here $a_n = \sin \frac{n\pi}{2} + \frac{(-1)^n}{n}$, $n \in \mathbb{N}$.

Since $-\frac{4}{3} \leq a_n \leq 1 \quad \forall n \in \mathbb{N}$, then the sequence is bounded, and the terms of the sequence are given by

$$a_{2n} = \frac{1}{2n} \text{ for all } n \in \mathbb{N}.$$

$$a_{4n+1} = 1 - \frac{1}{4n+1}, \quad \forall n \in \mathbb{N}$$

$$a_{4n+3} = -1 - \frac{1}{4n+3}, \quad \forall n \in \mathbb{N}$$

Therefore, $\bar{A}_n = 1$ for all n , and hence

$$\limsup a_n = 1.$$

Also, \underline{A}_n is calculated as follows:

$$\underline{A}_{4n} = -1 - \frac{1}{4n+3} \quad \forall n \in \mathbb{N}.$$

$$\underline{A}_{4n+1} = -1 - \frac{1}{4n+3} \quad \forall n \in \mathbb{N}$$

$$\underline{A}_{4n+2} = -1 - \frac{1}{4n+3}, \quad \forall n \in \mathbb{N}.$$

...

Hence $\liminf a_n = -1$.

Theorem: If $\{a_n\}$ is any sequence, then

$\liminf a_n = -\infty$ iff $\{a_n\}$ is not bounded below

$\limsup a_n = \infty$ iff $\{a_n\}$ is not bounded above.

Corollary: If $\{a_n\}$ is any sequence, then

$-\infty < \liminf a_n \leq +\infty$ iff $\{a_n\}$ is bounded below

$-\infty \leq \limsup a_n < +\infty$ iff $\{a_n\}$ is bounded above.

Theorem: A real number a is the limit inferior of a bounded sequence $\{a_n\}$ iff for each $\epsilon > 0$, the followings hold:

- (i) $a_n < a + \epsilon$, for infinitely many values of n
 (ii) there exists a positive integer m such that
 $a_n > a - \epsilon$, for all $n \geq m$.

Proof. Since $\{a_n\}$ is bounded, then $\liminf a_n$ is finite.

Let $\epsilon > 0$ be given. Then

$a = \liminf a_n = \sup_n A_n$
 which holds if and only if

$A_n \leq a$ for all n ,

$\exists m \in \mathbb{N}$ such that $A_m > a - \epsilon$.

if and only if

$\inf \{a_n, a_{n+1}, a_{n+2}, \dots\} \leq a$ for all n ,

$\exists m \in \mathbb{N}$ such that $\inf \{a_m, a_{m+1}, \dots\} > a - \epsilon$

if and only if

$a_n < a + \epsilon$ for infinitely many values of n ,

$\exists m \in \mathbb{N}$ such that $a_n > a - \epsilon$, $\forall n \geq m$,

that is, (i) and (ii) holds.

The theorem is proved.

Corollary: If $\{a_n\}$ is a bounded sequence and $\liminf a_n = a$, then

(i) for each real number $\alpha < a$, \exists a true integer m such that $a_n > \alpha$ $\forall n \geq m$; and

(ii) if $\alpha \in \mathbb{R}$ and $\exists m \in \mathbb{N}$ such that $a_n > \alpha$ $\forall n \geq m$, then $\liminf a_n \geq \alpha$.

 Hint: $\epsilon = a - \alpha$ in the first part of the above theorem to prove (i) of Corollary.

Theorem. A real number \bar{a} is the limit superior of a bounded sequence $\{a_n\}$ iff for each $\epsilon > 0$, the following hold:

- (i) $a_n > \bar{a} - \epsilon$, for infinitely many values of n .
 (ii) there exists a positive integer m such that $a_n < \bar{a} + \epsilon$, for all $n > m$.

Corollary of $\{a_n\}$ is a bounded sequence and $\bar{\lim} a_n = \bar{a}$, then

- (i) for each real number $\beta > \bar{a}$, $\exists m \in \mathbb{N}$ such that $a_n < \beta \quad \forall n > m$; and
 (ii) if $\beta \in \mathbb{R}$ and $\exists m \in \mathbb{N}$ such that $a_n < \beta \quad \forall n > m$, then $\bar{\lim} a_n \leq \beta$.

Corollary. Let $\{a_n\}$ be a bounded sequence and $\beta \in \mathbb{R}$.

Then

- (i) if $\bar{\lim} a_n > \beta$, then $\{n; a_n > \beta\}$ is infinite; and
 (ii) if $\{n; a_n > \beta\}$ is infinite, then $\bar{\lim} a_n \geq \beta$.

Theorem: If $\{a_n\}$ is a bounded sequence, then

- Snyp.
 (i) $\underline{\lim} a_n =$ smallest limit point of $\{a_n\}$
 (ii) $\bar{\lim} a_n =$ greatest limit point of $\{a_n\}$.

Note: The above theorem shows that the limit inferior and superior are the smallest and the greatest limit points, respectively of a bounded sequence, and hence the lower and upper limits of the considered bounded sequence.

Convergent Sequences.

A sequence may have unique, no, or multiple limit points.
Our interest: A bounded sequence with a unique limit point.

Obviously, such a sequence can have at the most a finite number of terms outside the interval $(l-\epsilon, l+\epsilon)$, $\epsilon > 0$, however small ϵ . For otherwise, by Bolzano-Weierstrass theorem, the infinite number of outside terms will have another limit point, further, the condition automatically ensures the existence of infinite no. of terms of the sequence within the interval. Such sequences are called convergent sequences.

Theorem Every bounded sequence with a unique limit point is convergent.

Proof. By Bolzano-Weierstrass theorem, the bounded sequence $\{s_n\}$ has a limit point. Let it be l . Then, for $\epsilon > 0$, $s_n \in (l-\epsilon, l+\epsilon)$ for an infinite no. of values of $n \in \mathbb{N}$.

l being the only limit point of the sequence, there can exist only a finite values, ~~of m~~ say m_1, m_2, \dots, m_r of n such that the corresponding terms of the sequence do not belong to $(l-\epsilon, l+\epsilon)$, because infinitely many outside terms will have another limit point which contradicts the assumption of the theorem.

Let $m-1$ be the greatest of such exceptional values of n . Then, we have $s_n \in (l-\epsilon, l+\epsilon)$ for all $n \geq m$, that is, $|s_n - l| < \epsilon$ for all $n \geq m$. Thus, the sequence converges to its unique limit point l . The theorem is proved.

Then, from the theorem

"Every convergent sequence is bounded and has a unique limit"

and

"Every bounded sequence with a unique limit point is convergent",

we state the following theorem:

Theorem A necessary and sufficient condition for the convergence of a sequence is that it is bounded and has a unique limit point.



In view of the above discussion, we give another definition of convergent sequence.

Definition. A sequence is said to be convergent if it is bounded and has a unique limit point.

Theorem. A necessary and sufficient condition for a sequence $\{s_n\}$ to converge to l is that for each $\epsilon > 0$, there corresponds a positive integer m such that

$$|s_n - l| < \epsilon \quad \forall n \geq m.$$

Theorem: A bounded sequence $\{a_n\}$ converges to a real number l if and only if

$$\underline{\lim} a_n = \overline{\lim} a_n = l.$$

Non-Convergent Sequences :

(a) Bounded Sequences :

A bounded sequence which does not converge and has at least two limit points, is said to ~~be~~ oscillate finitely.

(b) Unbounded Sequences :

(i) If a sequence $\{s_n\}$ is unbounded on the left (below), then we say that $-\infty$ is the limit point of the sequence, and to each positive number G , however large, there corresponds a positive integer m , such that

$$s_n < -G \quad \forall n \geq m,$$

~~that~~ i.e., the sequence has infinitely many terms below $-G$. So $-\infty$ is the limit point so that $\lim s_n = -\infty$.

(ii) If a sequence $\{s_n\}$ is unbounded on the right (above), then we say that $+\infty$ is the limit point of the sequence, and to each positive number G , there corresponds a positive integer m , such that

$$s_n > G \quad \text{for } n \geq m,$$

i.e., the sequence has an infinitely many terms above G . So $+\infty$ is the limit point so that

$$\lim s_n = \infty.$$

(iii) If a sequence $\{s_n\}$ is bounded above (on the right) but not below and besides $-\infty$, has no other limit points, then $-\infty$ is not only its least but also its greatest limit point.
Hence

$$\underline{\lim} S_n = \overline{\lim} S_n = \lim S_n = -\infty.$$

The sequence is said to diverge to $-\infty$.

(iv) If, finally, the sequence is bounded on the left (below) but not on the right (above) and besides ∞ , has no other limit points, then ∞ is not only the greatest but its least limit point. So we have

$$\overline{\lim} S_n = \underline{\lim} S_n = \lim S_n = \infty.$$

A sequence is then, said to diverge to ∞ ,

(v) An unbounded sequence is said to oscillate if it diverges neither to $+\infty$ nor to $-\infty$. A bounded sequence is either converge or else oscillate finitely, but an unbounded sequence either diverges to $+\infty$ or $-\infty$ or oscillates infinitely.

Illustrations:

- (i) $\{1 + (-1)^n\}$ oscillates finitely.
- (ii) $\{(-1)^n(1 + \frac{1}{n})\}$ oscillates finitely/infinitely? Verify.
- (iii) $\{n^2\}$ diverges to ∞ .
- (iv) $\{-2^n\}$ diverges to $-\infty$.
- (v) $\{n(-1)^n\}$ oscillates infinitely.
- (vi) $\left\{\frac{(-1)^{n-1}}{n!}\right\}$ converges to the limit 0.
- (vii) $\left\{1 + \frac{1}{n}\right\}$ converges to the limit 1.
- (viii) $\{1, 2, \frac{1}{2}, 3, \frac{1}{3}, \dots\}$ is bounded below but unbounded above, and has a limit point 0 besides $+\infty$.
 $\underline{\lim} S_n = 0, \overline{\lim} S_n = \infty.$

The sequence oscillates infinitely.

(IX) $\{ 1, 2, 3, 2, 5, 2, 7, 2, 3, 2, 11, 2, 13, \dots \}$,

$$S_n = \begin{cases} 2 & n \text{ even} \\ \text{lowest prime factor} & \text{when } n \text{ odd} \\ (\neq 1) \text{ of } n \end{cases}$$

n is bounded on the left but not on the right. It has infinite no. of limit points $2, 3, 5, 7, 11, \dots$ so that

$$\underline{\lim} S_n = 2, \quad \overline{\lim} S_n = \infty.$$

The sequence oscillates infinitely.

(X) The sequence $\{ m + \frac{1}{n} \}$, where m, n are natural numbers, also oscillates infinitely, $1, 2, 3, \dots$ being its limit points.

An Example (to open your eyes, how to choose ϵ, m etc...).

$$\text{Show that } \lim_{n \rightarrow \infty} \frac{3 + 2\sqrt{n}}{\sqrt{n}} = 2.$$

Let $\epsilon > 0$ be any positive number. We need to show/find m (depending on ϵ) such that

$$\left| \frac{3 + 2\sqrt{n}}{\sqrt{n}} - 2 \right| < \epsilon$$

$$\text{i.e. } \left| \frac{3}{\sqrt{n}} \right| < \epsilon, \text{ i.e., } n > \frac{9}{\epsilon^2} \quad (\text{choose } m)$$

Let m be a positive integer greater than $\frac{9}{\epsilon^2}$.

Then, for every $\epsilon > 0$, \exists a positive integer m such that

$$\left| \frac{3 + 2\sqrt{n}}{\sqrt{n}} - 2 \right| < \epsilon \quad \forall n \geq m.$$

Hence $\lim_{n \rightarrow \infty} \frac{3 + 2\sqrt{n}}{\sqrt{n}} = 2$. (proved).

Cauchy's General Principle of Convergence:

A necessary and sufficient condition for the convergence of a sequence $\{S_n\}$ is that, for each $\epsilon > 0$ there exists a positive integer m such that

$$|S_{n+p} - S_n| < \epsilon \quad \forall n \geq m \text{ and } p \geq 1.$$

Remark: Our earlier theorems can be used to test the convergence of a sequence to a limit l . However, if a limit l is not known, then Cauchy's general principle of convergence is used which involves only the terms of the sequence to determine a sequence is *convergent* or *divergent* etc...

Cauchy sequence:

A sequence $\{S_n\}$ is called a Cauchy sequence or a fundamental sequence if for each $\epsilon > 0$, there exists a positive integer m such that

$$|S_{n+p} - S_n| < \epsilon \quad \forall n \geq m \text{ and } p \geq 1$$

$$\& \quad |S_{n_1} - S_{n_2}| < \epsilon \quad \forall n_1, n_2 \geq m.$$

Remark: Thus, in the field of real numbers, a sequence is convergent iff it is a Cauchy sequence.

A sequence can not ~~converge~~ converge if even one $\epsilon > 0$ can be found such that for every positive integer m such that

$$|S_{n+p} - S_n| \not< \epsilon \quad \text{for } n \geq m \text{ and } p \geq 1.$$

Ex $\{a_n\}, \{b_n\} \rightarrow$ Cauchy sequences. Then $\{a_n \pm b_n\}$, $\{a_n b_n\} \rightarrow$ Cauchy sequences, if $b_n \neq 0 \forall n$, then $\{a_n/b_n\} \rightarrow$ Cauchy sequence.

This example helps us to determine/study the algebra of sequences.

H.W Show that a Cauchy sequence is bounded. ~~However~~
Verify its converse.

Ex. Show that the sequence $\{S_n\}$ where

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

cannot converge.

Pf/: Suppose, on the contrary that $\{S_n\}$ is convergent.
Set $\epsilon = 1/2$, and $n = m$ and $p = m$ in the Cauchy's
sequence general principle of convergence so that

$$|S_{2m} - S_m| < \epsilon = 1/2.$$

$$\text{But } S_{2m} - S_m = \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{2m}$$

$$> \frac{1}{2m} + \frac{1}{2m} + \dots + \frac{1}{2m} = \frac{m}{2m} = 1/2,$$

which is a contradiction. Hence the sequence cannot
converge.

Cauchy's first theorem on limits

$$\text{If } \lim_{n \rightarrow \infty} a_n = l, \text{ then } \lim_{n \rightarrow \infty} \left(\frac{a_1 + a_2 + \dots + a_n}{n} \right) = l.$$

Ex Show that

$$\lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \right] = 1.$$

Pf/: Let $a_n = \frac{n}{\sqrt{n^2+k}}$, $k = 1, 2, \dots, n$.

$$\text{Then } \lim_{n \rightarrow \infty} a_n = \frac{1}{\sqrt{1 + \frac{1}{n}}} = 1$$

By Cauchy's first theorem on limits, $\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \dots + a_n}{n} = 1$

$$\sim \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{n}{\sqrt{n^2+1}} + \frac{n}{\sqrt{n^2+2}} + \dots + \frac{n}{\sqrt{n^2+n}} \right] = 1$$

$$\sim \lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \right] = 1.$$

p.w.

Ex Show that $\lim_{n \rightarrow \infty} \frac{1}{n} (1 + 2^{1/2} + 3^{1/3} + \dots + n^{1/n}) = 1$.

Pf: Hint: $a_n = n^{1/n} \Rightarrow \lim a_n = 1$.

Use Cauchy's first theorem on limits to prove.

Cauchy's second theorem on limits

If all the terms of a sequence $\{a_n\}$ are positive and if $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ exists, then so does $\lim_{n \rightarrow \infty} a_n^{1/n}$, and two limits are equal, i.e.

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} a_n^{1/n}$$

provided that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ exists.

Ex Show that the sequence $\left\{ \frac{(3n)!}{(n!)^3} \right\}$ converges and find its limit.

Ans. $a_n = \frac{(3n)!}{(n!)^3} \Rightarrow a_{n+1} = \frac{(3n+3)!}{((n+1)!)^3}$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \dots = \lim_{n \rightarrow \infty} \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3} = 27.$$

By Cauchy's ^{second} Theorem on limits,

$$\lim_{n \rightarrow \infty} a_n^{1/n} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 27.$$

Theorem If $\{a_n\}$ is a sequence such that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = l$, where $|l| < 1$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Monotonic Sequences

A sequence $\{s_n\}$ is said to be monotonic increasing if $s_{n+1} \geq s_n \forall n$, and monotonic decreasing if $s_{n+1} \leq s_n \forall n$.

A sequence is strictly increasing if $\forall n, s_{n+1} > s_n$ and strictly decreasing if $\forall n, s_{n+1} < s_n$.

Theorem A necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.

Corollary A monotonic increasing bounded above sequence converge to its least upper bound and a monotonic decreasing bounded below to the greatest lower bound.

Corollary A monotonic increasing sequence which is not bounded above, diverges to $+\infty$.

Subsequences

If $\{s_n\} = \{s_1, s_2, \dots\}$ be a sequence, then any finite succession of its terms, picked out in any way (but preserving the original order), is called a subsequence.

In other words, if $\{n_k\}$ is a strictly increasing sequence of natural numbers, i.e., $n_1 < n_2 < n_3 < \dots$, then $\{s_{n_k}\}$ is a subsequence of the sequence $\{s_n\}$.

Illustrations

A sequence

1. $\{s_2, s_4, s_{10}, \dots, s_{n^2}\}$ is a subsequence of $\{s_n\}$.
2. $\{s_1, s_4, s_9, \dots, s_{n^2}, \dots\}$ is a subsequence of $\{s_n\}$.

Theorem Φ . A sequence $\{s_n\}$ converges to L iff its every subsequence converges to L . Similarly, $\lim s_n = \infty (-\infty)$ iff every subsequence of $\{s_n\}$ tends to $\infty (-\infty)$.

Theorem. If η is a limit point of a sequence $\{s_n\}$, then \exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ which converges to η , i.e.,

$$\lim_{k \rightarrow \infty} s_{n_k} = \eta.$$

Ex. The subsequence $\{n, n^4, n^9, n^{16}, \dots, n^{k^2}, \dots\}$ of $\{n^k\}$ converges to zero if $|n| < 1$, because the sequence $\{n^k\}$ converges to zero if $|n| < 1$.

Nested Intervals

If a sequence of closed intervals $[a_n, b_n]$ is such that each member $[a_{n+1}, b_{n+1}]$ is contained in the preceding one $[a_n, b_n]$ and $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$, then there is one and only one point common to all the intervals of the sequence.