Dynamics of Models with Allee Effects

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Definition of Allee effect

- Allee effects generally refer to a reduction in individual fitness at low population size or density.
- The Allee effect is an ecological concept with roots that go back at least to 1920. Any ecological mechanism that can lead to a positive relationship between a component of individual fitness and either the number or density of conspecifics can be termed as a mechanism of Allee effect or depensation or negative competition effect.
- Allee effect may arise from a number of sources such as genetic drift or inbreeding,
  difficulties in finding mates,
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Different Types of Allee effects

- There are different types of Allee effects: Additive Allee effect, Strong Allee effect, weak Allee effect and etc.
- Strong Allee effect leads to threshold population density, below which the population growth is negative and the population is likely to go extinct.
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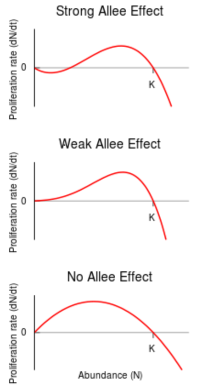
Pictorial Representation of Different Allee effects:



Different Allee effects

- Allee effects are classified by the nature of density dependence at low densities. If the population shrinks for low densities, there is a strong Allee effect. If the proliferation rate is positive and increasing then there is a weak Allee effect. The null hypothesis is that proliferation rates are positive but decreasing at low densities.
- Strong Allee effect: is an effect with a critical population size or density
- Weak Allee effect: is an Allee effect without a critical population size or density

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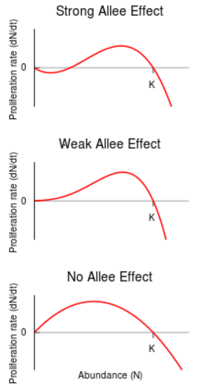

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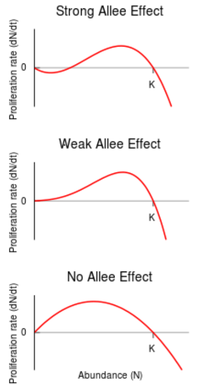

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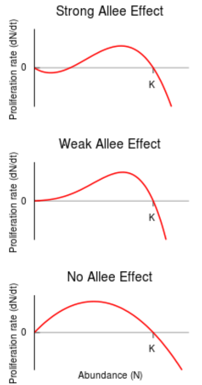

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Strong Allee effects

Strong Allee effect...................

Dynamic Equation involving Strong Allee Effect

- Dynamics of a Population influenced by strong Allee effect are described by

$$\frac{dy}{dt} = ay(y-b)(c-y), a > 0, 0 < b < c \tag{1}$$

where the constants

a represents intrinsic growth rate of the population

b represents the threshold value below which the growth rate is negative

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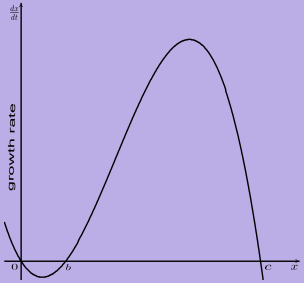

- In this figure the growth rate is non positive in the interval $[0, b]$ and positive in the interval (b, c).
- Thus, if the initial population $y(0) \in (0, b)$ the population goes extinct due to negative growth rate.
- If the initial population $y(0) > b$, then the population eventually reaches its carrying capacity c.
- It is well known that the equation (1) admits two positive equilibrium solutions given by $y(t) = b$ and $y(t) = c$ and one trivial equilibrium solution.

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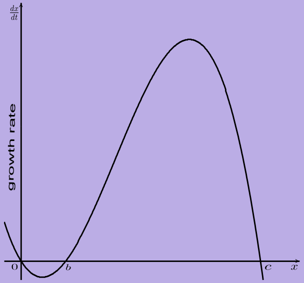

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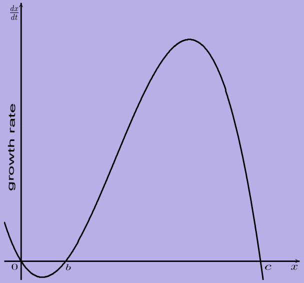

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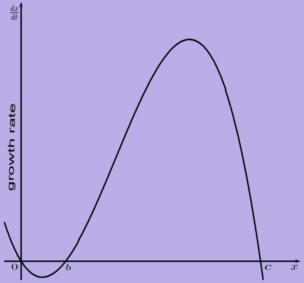

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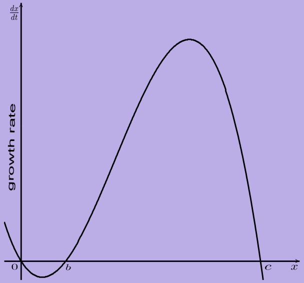

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Periodic environment

- We know that the conditions in the environment are driven by seasons.
- The birth and death rates, carrying capacity also depend on the seasons.
- Hence it is more appropriate to consider the population dynamics in a seasonally varying environment.
- Seasonality can be incorporated into the governing dynamic equations by taking the involved coefficients to be periodic.

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Periodic environment

- We introduce seasonality into the dynamics of the renewable resource by assuming the coefficients a, b, c to be nonnegative T-periodic functions of same period.
- Thus the revised model is a periodic differential equation given by

$$\frac{dy}{dt} = a(t)y(y - b(t))(c(t) - y) \tag{2}$$

  $a(t)$ represents seasonal dependent intrinsic growth rate of the resource

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Model analysis: $\frac{dy}{dt} = a(t)y(y - b(t))(c(t) - y)$

- Consider the transformation

$$y(t) = c(t)x(t)$$

then the considered model gets transformed to

$$\frac{dx}{dt} = -\left(a(t)c^2(t)k(t) + \frac{c'(t)}{c(t)}\right)x + a(t)c^2(t)\left((1 + k(t)) - x\right)x^2 \quad (3)$$

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- It is easy to observe that the above equation is a particular case of a general scalar differential equation of the form

$$\frac{dx}{dt} = -A(t)x(t) + f(t, x(t)) \tag{4}$$

where $A \in C(R, R), f \in C(R \times R, R)$ satisfying $A(t+T) = A(t), f(t+T, x) = f(t, x)$.

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- Application of Leggett-Williams multiple fixed point theorem on cones yields existence of at least two positive periodic solutions for this equation.
- These existence results are applied to the considered model to obtain information on existence of positive T-periodic solutions.

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Lemma

If $x(t)$ is a T - periodic solution of $\frac{dx}{dt} = -A(t)x(t) + f(t, x(t))$ then it satisfies the integral equation

$$x(t) = \int_t^{t+T} G(t,s) f(s, x(s)) ds \tag{5}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \frac{\exp(\int_t^s A(\theta)d\theta)}{\exp(\int_0^T A(\theta)d\theta) - 1}, t, s \in R. \tag{6}$$

Notations and Fundamentals

- Let us define

$$\delta = \exp\left(\int_0^T A(\theta)d\theta\right). \tag{7}$$

- Observe that $\delta > 1$ if

$$\int_0^T A(\theta)d\theta > 0. \tag{8}$$

- Under the assumption (8) the Green's function (6) satisfies

$$0 < \frac{1}{\delta - 1} < G(t,s) < \frac{\delta}{\delta - 1}, \quad s \in [t, t+T]. \tag{9}$$

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Notations and Fundamentals

- Let X be a Banach space and K be a cone in X. A mapping $\psi : K \to [0, \infty)$ is said to be a concave nonnegative continuous functional on K if it is continuous and satisfies

$$\psi(\eta x + (1-\eta)y) \geq \eta\psi(x) + (1-\eta)\psi(y), x, y \in K, \eta \in [0, 1].$$

- Let $a, b, c > 0$ be constants with K and X as defined above. Define

$$K_a = \{x \in K : \|x\| < a\} \tag{10}$$

and

$$K(\psi, b, c) = \{x \in K; \psi(x) \geq b, \|x\| < c\}. \tag{11}$$

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Leggett-Williams multiple fixed point theorem

Suppose $E : \overline{K}_{c_3} \to K$ is completely continuous, and suppose there exists a concave nonnegative functional ψ with $\psi(x) \leq \|x\|, \; x \in K$ and numbers c_1 and c_2, with $0 < c_1 < c_2 < c_3$ satisfying the following conditions:

- i $\{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \phi$ and $\psi(Ex) > c_2$ if $x \in K(\psi, c_2, c_3)$
- ii $\|Ex\| < c_1$ if $x \in \overline{K}_{c_1}$ and
- iii $\psi(Ex) > \frac{c_2}{c_3}\|Ex\|$ for each $x \in \overline{K}_{c_3}$ such that $\|Ex\| > c_3$.

Then E has at least two fixed points in $\overline{K}_{c_3}$.

Lemma

The set

$$X = \{x \in C([0,T],R) : x(0) = x(T)\} \tag{12}$$

is a Banach space endowed with the norm

$$\|x\| = \sup_{0 \le t \le T} x(t) \tag{13}$$

where $C[0,T]$ is the set of all continuous functions defined on $[0,T]$.

Theorem

Let $\int_0^T A(s)ds > 0$. *Suppose*

(H_1) *there exists a positive constant* c_3 *such that* $\int_0^T f(s,x(s))ds > 0$ *when* $0 < x(s) \le c_3$ *for all* $s \in [0,T]$ *and*

$$\int_0^T f(s,x(s))ds \ge \frac{\delta-1}{\delta}c_3 \ \ \text{when} \ \ \frac{c_3}{\delta} \le x(s) \le c_3,$$

$$s \in [0,T]$$

(H_2) $\lim\limits_{\|x\|\to 0} \frac{1}{\|x\|} \int_0^T f(s,x(s))ds < \frac{\delta-1}{\delta}$

hold. Then the equation $\frac{dx}{dt} = -A(t)x(t) + f(t,x(t))$ *has at least two positive T-periodic solutions in* $\overline{K}_{c_3}$.

Corollary

Let $\int_0^T A(s)ds > 0$. *Suppose*

(H_1) *there exists a positive constant* c_3 *such that*

$$\int_0^T f(t,x)dt > 0 \quad \text{for} \quad 0 < x \le c_3.$$

Further, for the above choice of c_3, *assume that*

$$\int_0^T f(s,x)ds = \frac{\delta - 1}{\delta} x \quad \text{for} \quad x = c_3$$

$$\text{and} \quad \int_0^T f(s,x)ds > \frac{\delta - 1}{\delta} c_3 \quad \text{for} \quad \frac{c_3}{\delta} \le x < c_3 \tag{14}$$

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Application to renewable resource dynamics

- we shall apply the above results to investigate the existence of positive T- periodic solutions for the equation $\frac{dy}{dt} = a(t)y(y - b(t))(c(t) - y)$.
- Here we have

$$A(t) = \left(a(t)c^2(t)k(t) + \frac{c'(t)}{c(t)} \right) \tag{15}$$

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$$f(t,x) = a(t)c^2(t)\left((1 + k(t)) - x\right)x^2. \tag{16}$$

Application to renewable resource dynamics

- we shall apply the above results to investigate the existence of positive T- periodic solutions for the equation $\frac{dy}{dt} = a(t)y(y - b(t))(c(t) - y)$.
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- Let us denote

$$M = \int_0^T a(s)c^2(s)ds \text{ and } N = \int_0^T a(s)c^2(s)k(s)ds. \quad (17)$$

- Since $0 < k(t) < 1$, we have $M > N > 0$.
- Since $f(t,x) = a(t)c^2(t)\left((1+k(t)) - x\right)x^2$ $\lim\limits_{x\to 0} \frac{1}{x}\int_0^T f(s,x)ds = 0$ and hence H_2^* of Corollary 3 is satisfied by the considered equation (3).

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Theorem

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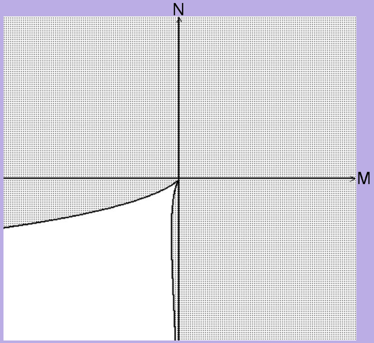
$$M = \int_0^T a(s)c^2(s)ds \text{ and } N = \int_0^T a(s)c^2(s)k(s)ds.$$

If

$$\frac{(M+N) + \sqrt{(M+N)^2 - 4M(\frac{e^N-1}{e^N})}}{2M} > \frac{e^{2N} - \frac{1}{e^N}}{M+N} \tag{18}$$

then (3) has at least two positive T-periodic solutions.

$$\frac{(M+N)+\sqrt{(M+N)^2-4M(\frac{e^N-1}{e^N})}}{2M} > \frac{e^{2N}-\frac{1}{e^N}}{M+N}$$



- The shaded region represents portion of the (M, N) space which satisfies $(M+N)^2 - 4M(\frac{e^N-1}{e^N}) > 0$.

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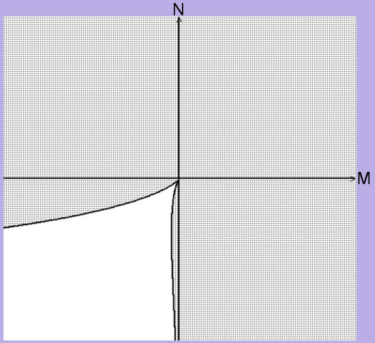

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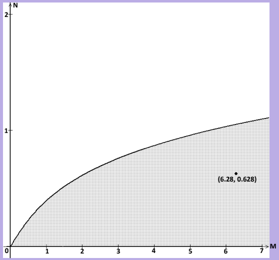
Numerical Simulation



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- Choose the functions $a(t), b(t)$ and $c(t)$ to be the following 2π periodic functions

$$a(t) = (1.2+\sin t)^2, \; b(t) = \frac{1.2+\cos t}{12(1.2+\sin t)} \text{ and } c(t) = \frac{1}{1.2+\sin t}.$$

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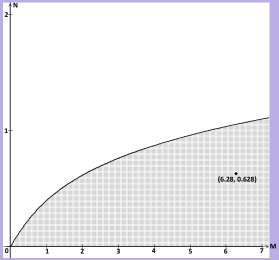

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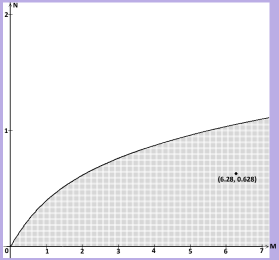

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Existence of exactly two periodic solutions

- The model

$$\frac{dy}{dt} = a(t)y(y - b(t))(c(t) - y)$$

- This can be written as

$$\frac{dy}{dt} = a(t)y(-y^2 + (b(t) + c(t))y - b(t)c(t))$$

- Since $a(t)$,$b(t) + c(t)$ and $b(t)c(t)$ are positive T-periodic functions, there exist positive constants L_1, L_2, M_1, M_2, N_1 and N_2 such that

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$$\frac{dy}{dt} = yF(t,y), \quad \frac{dy}{dt} = yG(y) \text{ and } \frac{dy}{dt} = yH(y). \tag{19}$$

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$$\begin{aligned} F(t,y) &= a(t)(-y^2 + (b(t)+c(t))y - b(t)c(t)), \\ G(y) &= L_1(-y^2 + M_1 y - N_2) \text{ and} \\ H(y) &= L_2(-y^2 + M_2 y - N_1). \end{aligned}$$

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Theorem

If a solution $\phi(t,0,x_0)$ of a T-periodic differential equation $\dot{x} = f(t,x)$ is bounded for $t \geq 0$, then there is a T-periodic solution $\Phi(t)$ of equation $\dot{x} = f(t,x)$ such that

$$\phi(t+Tk,0,x_0) \to \Phi(t) \ \text{ as the integer } \ k \to \infty$$

monotonically and uniformly for $0 \leq t \leq T$.

Similarly, if $\phi(t,0,x_0)$ is bounded for $t \leq 0$, then there is a T-periodic solution $\Psi(t)$ of equation $\dot{x} = f(t,x)$ such that

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Theorem

Let $\dot{x} = xf(t,x)$ be a T-periodic differential equation. Let there exist a function $g(x)$ such that $g(x) < f(t,x)$ for all t. Let $x^* > 0$ be such that $g(x^*) = 0$ and $f(t,x)$ be decreasing in x for $x > x^*$ and all t. Let $M > x^*$ such that $f(t,x) < 0$ for $x > M$ and all t. Then

(a) *if $x(0) \geq x^*$, then $x(t) \geq x^*$ for all t*

(b) *there is a unique positive $T-$ periodic solution to which any other solution with $x(0) > x^*$ approaches.*

Theorem

Assume that $M_1^2 - 4N_2 > 0$. If $M_2(2M_1 - M_2) > 4N_2$ then the equation (2) has exactly two positive T-periodic solutions given by $z_T(t)$ and $y_T(t)$ with $z_T(t) < y_T(t)$ such that $y_-^* \leq z_T(t) \leq y_-$ and $y_+ \leq y_T(t) \leq y_+^*$ where y_-, y_+, y_-^* and y_+^* are positive equilibrium solutions of $\frac{dy}{dt} = yH(y)$ and $\frac{dy}{dt} = yG(y)$ respectively.

Further more, the trivial solution and $y_T(t)$ are locally asymptotically stable and $z_T(t)$ is unstable. All solutions initiating below $z_T(t)$ eventually approach the zero solution while those initiating above $z_T(t)$ reach $y_T(t)$ eventually.

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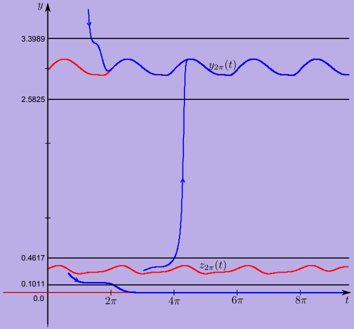

- Among the two periodic solutions, the one with smaller magnitude is unstable and the remaining is asymptotically stable.
- The unstable periodic solution separates the regions of attractions of the two asymptotically stable periodic solutions in the positive quadrant.

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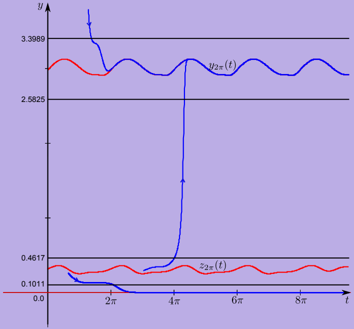

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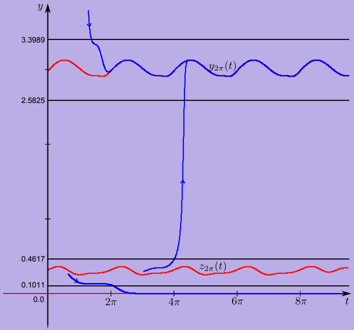

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Additive Allee effects

Additive Allee Effects...........

Dynamic Equation involving Additive Allee Effect

- Dynamics of a Population influenced by Additive Allee effect are described by

$$\frac{dy}{dz} = ry\left(1 - \frac{y}{K} - \frac{\gamma}{1+wy}\right), \tag{21}$$

Here the positive constants r, K respectively, represent intrinsic growth rate and carrying capacity of the resource γ and w indicate the severity of Allee effect.

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Dynamic Equation involving Additive Allee Effect: Periodicity

- Now, we focus on the study of the qualitative behaviour of (21) under the assumption that the associated parameters are periodic of the same period. Hence, it is assumed that the coefficients in (21) to be nonnegative P-periodic continuous functions, given by $r(\tau)$, $K(\tau)$, $\gamma(\tau)$ and $w(\tau)$. Under this assumption, (21) gets modified to

$$\frac{dy}{d\tau} = r(\tau)y\left(1 - \frac{y}{K(\tau)} - \frac{\gamma(\tau)}{1 + w(\tau)y}\right) \tag{22}$$

- The transformation $t = G(\tau) = \int_0^\tau r(s)\,ds$ transforms the equation (22) to a T-periodic equation given by

$$\frac{dx}{dt} = x\left(1 - \frac{x}{K(t)} - \frac{\eta(t)}{1 + m(t)x}\right) \tag{23}$$

with $x(t) = y(G^{-1}(t))$, where $\eta(t) = \gamma(G^{-1}(t))$, $m(t) = w(G^{-1}(t))$ and $K(t) = K(G^{-1}(t))$ are positive periodic functions of period $T = G(P)$. Also, for each T-periodic solution $x(t)$ of (23), $y(\tau) = y(G^{-1}(t))$ defines a P-periodic solution of (22).

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Dynamic Equation involving Additive Allee Effect: Periodicity

- The technique used in this model: the method of upper and lower solutions. The interesting character of this case is that we have obtained the dynamics of the positive T-periodic solutions while applying upper and lower solution method.

Dynamic Equation involving Additive Allee Effect: Main result

- Since the coefficients $K(t)$, $\eta(t)$ and $m(t)$ of (23) are positive, T-periodic continuous functions, they are bounded and hence, there exist positive constants k_1, k_2, a, b, c and d such that

$$k_1 \leq K(t) \leq k_2,$$

$$a \leq \eta(t) \leq b$$

and

$$c \leq m(t) \leq d.$$

Theorem:

Suppose that $a > 1$, $ck_1 > 1$ and $(ck_1 - 1)^2 + 4ck_1(1 - b) > 0$ holds. Then we have the following:

- Equation (23) has a unique asymptotically stable nontrivial periodic solution $x_1^*(t)$ such that

$$\frac{ck_1 - 1 + \sqrt{(ck_1 - 1)^2 + uck_1(1-b)}}{2c}$$
$$< x_1^*(t) <$$
$$\frac{dk_2 - 1 + \sqrt{(dk_2 - 1)^2 + 4dk_2(1-a)}}{2d};$$

- Equation (23) has an unstable periodic solution $x_2^*(t)$ such that

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Dynamic Equation involving Additive Allee Effect: Example

Consider the equation

$$x' = x\left(1 - \frac{x}{(1+\frac{1}{5}\sin t)} - \frac{(2.06+\cos t)}{1+x(16.6+\sin t)}\right). \qquad (24)$$

Clearly, $K(t) = 1 + \frac{1}{5}\sin t$, $\eta(t) = 2.06 + \cos t$ and $m(t) = 16.6 + \sin t$. Then $k_1 = 0.8$, $k_2 = 1.2$, $a = 1.06$, $b = 3.06$, $c = 15.6$ and $d = 17.6$. Observe that $a > 1$ and $ck_1 = 12.48 > 1$. Further,

$$\frac{ck_1 - 1 - \sqrt{(ck_1-1)^2 + 4ck_1(1-b)}}{2c} = \frac{6.0989963761}{31.2} = 0.1954806531,$$

$$\frac{ck_1 - 1 + \sqrt{(ck_1-1)^2 + 4ck_1(1-b)}}{2c} = \frac{16.8610036239}{31.2} = 0.5404167828,$$

$$\frac{dk_2 - 1 - \sqrt{(dk_2-1)^2 + 4dk_2(1-a)}}{2d} = \frac{0.1263610116}{35.2} = 0.0035898015$$

and

$$\frac{dk_2 - 1 + \sqrt{(dk_2-1)^2 + 4dk_2(1-a)}}{2d} = \frac{40.1136389884}{35.2} = 1.1395920167$$

shows that the conditions of the above theorem are satisfied. Hence the equation (24) has an asymptotically stable periodic solution $x_1^*(t)$ satisfying the bounds

$$0.5404167828 < x_1^*(t) < 1.1395920167,$$

an unstable solution $x_2^*(t)$ with the bounds

$$0.0035898015 < x_2^*(t) < 0.1954806531,$$

and a trivial solution $x_3^*(t)$, which is asymptotically stable.

Dynamic Equation involving Additive Allee Effect: Numerical treatment..

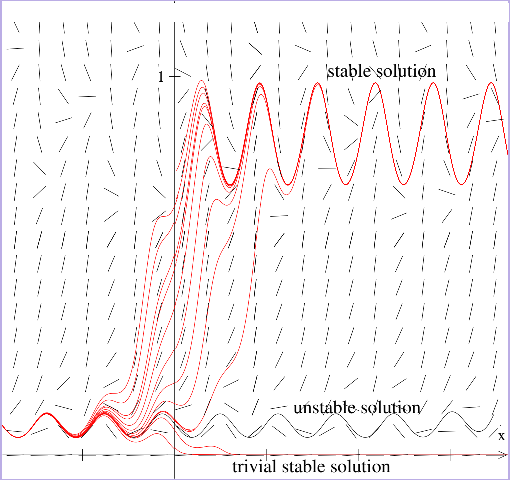
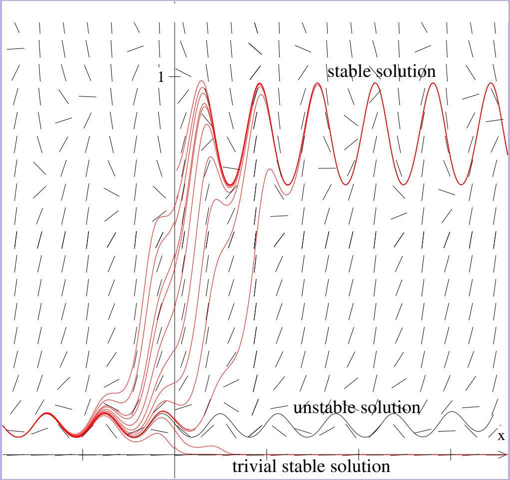


Fig.1: Computer simulation of the above Example

Dynamic Equation involving Additive Allee Effect: Numerical treatment..



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Allee effects: Physical Interpretation of solution properties:

Consider the Scalar Delay Differential equation: Hematopoiesis model with Strong Allee

$$x'(t) = \frac{r(t)x(t)}{1 + x^{\gamma}(t)} - b(t)x(t) - a(t)x(g(t)), \tag{25}$$

with the initial data $x(t) = \phi(t), t < 0, x(0) = x_0$, $\gamma > 0$, $r(t) \geq 0, b(t) \geq 0, a(t) > 0$ are bounded functions on $[0, \infty)$ (obviously, holds for periodic functions), $g(t) \leq t, g(t) \to \infty$ as $t \to \infty$, $\phi(t) > 0$ and $x_0 > 0$. Here $r \to$ fecundity rate, $a \to$ hunting rate, $b \to$ mortality rate and $\gamma \to$ abruptness parameter.

Theorem: Physical Interpretation:

- **Statement of the Theorem:** Under the above conditions, Equation (25) has a unique local positive solution. This solution either becomes negative or is a global positive bounded solution.
- **Physical Interpretation:** The problem is well posed, as justified in the above theorem. It either has a positive solution for all t which is bounded (the size of the population cannot infinitely grow due to a negative feedback which is a typical situation in population dynamics) or becomes negative in some finite time.

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Theorem: Physical Interpretation:

- **Statement of the Theorem:** Let $0 < \phi(t) \leq x_0$, $A_0 = \sup_{t \geq 0} x(t)$, and satisfies either

$$\sup_{t \geq 0} \int_{g(t)}^{t} a(s) \exp\left(- \int_{g(s)}^{s} \left[\frac{r(\theta)}{1 + A_0^{\gamma}} - b(\theta) \right] d\theta \right) ds \leq \frac{1}{e}, \tag{26}$$

or

$$\sup_{t \geq 0} \int_{g(t)}^{t} [a(s) + b(s)]\, ds \leq \frac{1}{e}$$

. Then every solution of (25) is bounded by A_0.

Theorem: Physical Interpretation:

- **Physical Interpretation:** This theorem provides sufficient conditions for the positiveness of solutions and presents an upper bound for a solution. Inequality $0 < \phi(t) \leq x_0$ is vital for nonextinction in the following sense. If the harvesting rate is based on the size of the population some time ago, then for the survival of the population it is important that the field data on the population size is collected at the time when the population is not abundant. If the initial value is less than the initial function, then the harvesting based on the oversized estimation of the population can lead to the extinction at the very beginning of the history (when the influence of the prehistory is still significant).

Theorem: Physical Interpretation:Contd...

- **Physical Interpretation:** In (26), sufficient condition on harvesting, mortality and growth rates, and the delay provide the solution is positive. The greater the mortality and the harvesting rates are, the smaller should be delays providing that there is no extinction of the population in some finite time. Or else, for prescribed delays and a given natural growth rate r(t) and the mortality rate b(t) the harvesting rate should not exceed a certain number to avoid possible extinction. As the growth rate r(t) becomes higher, the greater can be the allowed delay in harvesting.

Theorem: Physical Interpretation:

- **Statement of the Theorem:** Assume that $\int_0^\infty [a(t) + b(t) - r(t)]\, dt = \infty$ and one of the following conditions hold:

$$b(t) \geq r(t)$$

or

$$a(t) + b(t) \geq r(t) \geq b(t), \; \limsup_{t \to \infty} [r(t) - b(t)](t - g(t)) < 1.$$

Then every solution of the equation (25) approaches to zero as $t \to \infty$.

- **Physical Interpretation:** This Theorem claims that if the total of the mortality and harvesting rates exceeds the birth rate, then the population is destined to extinct. It either equals to zero at some finite moment of time or tends to zero.

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Some References

- ALLEE, W.C., *Animal aggregations*, University of Chicago Press, Chicage, IL, 1931.
- COURCHAMP, F., CLUTTON-BROCK, T., GRENFELL, B., *Inverse density dependence and the Allee effect*, TREE, 14(10) (1999), pp. 405-410.
- COURCHAMP, F., BEREC, L., GASCOIGNE, J., *Allee effects in Ecology and Conservation*, Oxford University Press, New York, 2008.
- HALE, J. K., KOCAK, H., *Dynamics and Bifurcations*, Texts in Applied mathematics 3, Springer Verlag, New York, 1991.
- KING, A.C., BILLINGHAM, J., OTTO, S.R., *Differential Equations Linear, Nonlinear, Ordinay, Partial*, Cambridge University Press, USA, 2003.

Some References

- PADHI, S., SRINIVASU P.D.N. AND JOHN R. GRAEF, *Periodic Solutions of First order Functional Differential Equations in Population Dynamics*, Springer India, 2014.
- PADHI, S., SRINIVASU, P.D.N., KIRAN KUMAR, G., *Periodic Solutions for an equation governing dynamics of a Renewable Resource Subjected to Allee Effects*, Nonlinear Analysis : Real World Applications, 11 (2010), pp. 2610-2618.
- PONTRYAGIN, L.S., BOL'TYANKII, V.G., GAMKRELIDGE, R.V., MISHCHENKO, E.F., *The Mathematical Thoery of Optimal Processess*, pergamon press, New York, 1964.
- STEPHENS, P.A., SUTHERLAND, W. J., *Consequences of the Allee effect for behaviour, ecology and conservation*, TREE, 14(10) (1999), pp. 401-405.
- STEPHENS, P.A., SUTHERLAND, W.J., FRECKLETON, R.P., *What is the Allee effect?*, Oikos, 87 (1999), pp. 185-190.

THANK YOU