A Note on the Eigenvalue Criteria for Positive Solutions of a Cantilever Beam Equation with Free End

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Abstract. In this paper, we study the existence of at least one positive solution to the fourth-order two-point boundary value problem(BVP)

$$\begin{cases} u''''(t) = \lambda q(t) f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

which models a cantilever beam equation, where one end is kept free. Here $f \in \mathcal{C}([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$, $g \in \mathcal{C}([0,1], \mathbb{R}_+)$ and λ is a positive parameter. The sufficient conditions are interesting, new and easy to verify. We have used some inequalities on the nonlinear function f and eigenvalues of a linear integral operator as bounds for the parameter λ in order to obtain our results. Our approach is based on a revised version of a fixed point theorem due to Gustafson and Schmitt.

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1. Introduction

In this work, we are interested in demonstrating the use of revised versions of two fixed point theorems due to Gustafson and Schmitt [6, 7] for studying the existence of a positive solution to the nonlinear fourth order two point boundary value problem (BVP)

$$\begin{cases} u''''(t) = \lambda q(t) f(t, u(t)), \quad t \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{1.1}$$

This work was completed with the support of our TEX-pert.

which describes a cantilever beam equation on the deflection of an elastic beam fixed at left end and freed at the other end. Here we assume that $f \in \mathcal{C}([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$ and $q \in \mathcal{C}([0,1], \mathbb{R}_+)$ and λ is a positive parameter.

Boundary value problems of type (1.1) with various nonlinearities on f, have been studies by many authors [1, 2, 3, 5, 8, 9, 11, 12, 13, 17, 18, 20]. The methods used in these papers are contracting mapping principle, iterative method, fixed point index theory in cones, Krasnosel'skii fixed point theorem, lower and upper solution method and degree theory.

Our results in this paper are completely different from the approach by the authors in [1, 2, 3, 5, 8, 9, 11, 12, 13, 17, 18, 19, 20]. In a recent work [15], the author applied monotone iterative method to obtain sufficient conditions on the existence of one positive solution of (1.1), and an iterative scheme for approximating the solutions. In this present work, we shall use two fixed point theorems by Gatika and Smith [12] to provide ranges on the parameter λ in (1.1) in order to obtain sufficient conditions for the existence of positive solutions. The ranges on the parameter depends on the eigenvalues a linear integral operator, that is, (3.2) given in Section 3. Unlike the other papers, in this article, we shall study the existence of positive solutions of the integral equation

$$u(t) = \int_0^1 G(t,s)q(s)f(s,u(s))\,ds, \tag{1.2}$$

corresponding to the BVP (1.1), where $G(t,s)$ is a Green's function.

The motivation for this present study has come from the paper due to Webb and Lan [16], a recent work due to Cheng et al. [4] and a recent work due Padhi et al. [14]. Webb and Lan [16] studied the existence of positive solutions of (1.2) with an arbitrary and measurable Green's function $G(t,s)$ (See Remark 3.11 for the conditions on $G(t,s)$). Motivated by the work of Ituriiaga et al. [10], Padhi et al. [14] introduced a measurable function $b(t)$. The ranges on λ in our theorems are completely dependent on the first eigenvalue of the eigenvalue problem, considered in Section 3, with $m(t) = q(t)b(t)$. Most recently, Cheng et al. [4] studied the existence of positive solutions of a system of Hammerstein integral equations, which are the generalizations of (1.2), where the Green's function satisfies the condition conditions $G(t,s) = G(s,t)$, $s,t \in \Omega$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. In this paper, we have used the property $G(t,s) = G(s,t)$, $s,t \in [0,1]$ in [4], and techniques employed [14] and [16], and applied the eigenvalue and the corresponding eigenfunction of a linearized integral equation corresponding to (1.2). Thus, our results can be applied to those differential equations which can be transformed into an integral equation of the form (1.2).

Our results can also be extended to system of integral equations of the form

$$u_i(t) = \int_0^1 G_i(t,s)q(s)f_i(s,u_1(s),u_2(s),\cdots u_i(s))\,ds,\ i = 1,2,\cdots,n,$$

where $f_i \in \mathcal{C}\left([0,1] \times \mathbb{R}_+^n, \mathbb{R}_+\right),\ i = 1,2,\cdots,n$, and $q \in \mathcal{C}\left([0,1], \mathbb{R}_+\right)$ and $\lambda_i,\ i = 1,2,\cdots,n$ are positive parameters, $G_i(t,s) \geq 0$ for all $t, s \in [0,1],\ i = 1,2,\cdots,n$, and measurable, satisfying the property (3.3) for all $i = 1,2,\cdots,n$.

This work has been divided into three sections. Section 1 contains the basic informations on the BVP (1.1). Section 2 is Preliminary, where all basic results are incorporated. In Section 3, we prove the main results of this paper.

2. Preliminaries

We shall use the following fixed point results in a cone [7] that are revised versions of theorems due to Gustafson and Schmitt [6].

Theorem 2.1. *Let X be a Banach space and K be a cone in X. Let r and R be real numbers with $0 < r < R$,*

$$D = \{u \in K : r \leq \|u\| \leq R\},$$

and let $T : D \to K$ be a compact continuous operator such that

(a) *$u \in D$, $\mu < 1$, and $u = \mu T u$ imply $\|u\| \neq R$;*
(b) *$u \in D$, $\mu > 1$, and $u = \mu T u$ imply $\|u\| \neq r$;*
(c) $\inf\limits_{\|u\|=r} \|Tu\| > 0$.

Then T has a fixed point in D.

Theorem 2.2. *Let X be a Banach space and K be a cone in X. Let r and R be real numbers with $0 < r < R$,*

$$D = \{u \in K : r \leq \|u\| \leq R\},$$

and Let $T : D \to K$ be a compact continuous operator such that

(a) *$u \in D$, $\mu > 1$, and $u = \mu T u$ imply $\|u\| \neq R$;*
(b) *$u \in D$, $\mu < 1$, and $u = \mu T u$ imply $\|u\| \neq r$;*
(c) $\inf\limits_{\|u\|=R} \|Tu\| > 0$.

Then T has a fixed point in D.

In this paper, we set $X = \mathcal{C}[0,1]$ to be the Banach space with standard norm

$$\|u\| = \max_{0 \leq t \leq 1} |u(t)|. \tag{2.1}$$

Define a cone K on X by

$$K = \{u \in \mathcal{C}[0,1] : u(t) \geq 0, t \in [0,1]\}, \tag{2.2}$$

and an operator $T : K \to K$ by

$$Tu(t) = \lambda \int_0^1 G(t,s) q(s) f(s, u(s))\, ds, \tag{2.3}$$

where $G(t,s)$ is the Green's kernel, given by

$$G(t,s) = \frac{1}{6}\begin{cases} s^2(3t-s), & 0 \le s \le t \le 1, \\ t^2(3s-t), & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

Let $g(s) = \frac{s^2}{2}$ and $c(t) = \frac{2}{3}t^2$. From a straight forward calculation, also proved in [11], one can show that $G(t,s)$ satisfies the inequality

$$c(t)g(s) \le G(t,s) \le g(s) \text{ for } 0 \le t, s \le 1. \tag{2.5}$$

Since (2.5) is valid for any $t \in [0,1]$, then we have

$$\frac{1}{24} \cdot \frac{s^2}{2} := \frac{g(s)}{24} \le G(t,s) \le \frac{s^2}{2} := g(s) \tag{2.6}$$

replaces (2.5), where $\min_{t\in[1/4,3/4]} c(t) = \min_{t\in[1/4,3/4]} \frac{2}{3}t^2 = \frac{1}{24}$. Thus, we have the following important lemma.

Lemma 2.3. *$u(t)$ is a positive solution of the problem (1.1) if and only if $u(t)$ is a fixed point of the operator T on the cone K. Further, the solution $u(t)$ of (1.1) satisfies the inequality*

$$\min_{t\in[1/4,3/4]} u(t) \ge \frac{1}{24}\|u\|. \tag{2.7}$$

In order to satisfy condition (c) in Theorems 2.1 and 2.2, we shall make extensive use of the following lemma.

Lemma 2.4. *Let $f(t,u) > 0$ for $t \in (0,1)$ and $u > 0$. If $R > 0$ is a real number, then*

$$\inf\{\|Tu\| : u \in P,\ \|u\| = R\} > 0$$

for any solution u of (1.1).

Proof. Clearly, $u(t)$ in a solution of (1.1) if and only if $Tu = u$. Since $f(t,u) > 0$ for $t \in [\frac{1}{4}, \frac{3}{4}]$ and $u \in [\frac{1}{24}R, R]$, for

$$p = \inf\left\{ f(t,u) : (t,u) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{R}{24}, R\right] \right\} > 0$$

and

$$q = \inf_{t\in[\frac{1}{4},\frac{3}{4}]} t^2 q(t),$$

we have

$$(Tu)(t) = \lambda \int_0^1 G(t,s)q(s)f(s,u(s))\,ds \ge \lambda\frac{1}{48}\int_{\frac{1}{4}}^{\frac{3}{4}} s^2 q(s)f(s,u(s))\,ds \ge \frac{\lambda}{96}pq > 0,$$

and so $\|Tu\| \ge \frac{\lambda}{96}pq > 0$ for all $u \in K$ with $\|u\| = R$. This proves the lemma. $\square$

3. Main Results

In this section, we consider the operator T defined in (2.3), the Banach space X in (2.1), and the cone K in (2.2).

Let us consider the eigenvalue problem

$$\begin{cases} u''''(t) = q(t)b(t)u(t), \quad t \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{3.1}$$

where $b : [0,1] \to [0,\infty)$ is a continuous function. Let L be the linear operator defined by

$$Lu(t) = \int_0^1 G(t,s)q(s)b(s)u(s)\,ds, \tag{3.2}$$

where $G(t,s)$ is the Green's kernel, given (2.4). Then the existence of a positive solution of (3.1) is equivalent to the existence of a fixed point of the operator L in the cone K, defined in (2.2). The Green's function $G(t,s)$ is measurable, positive for all $t,s \in [0,1]$, and satisfies the property

$$\lim_{t\to\tau} |G(t,s) - G(\tau,s)| = 0 \ \mathit{for\ a.e.}\ s \in [0,1], \ \mathit{and}\ \tau \in [0,1]. \tag{3.3}$$

Applying the above, and assuming $\int_0^1 s^4 q(s)b(s)\,ds < \infty$, we can prove that $L : K \to K$ is a completely continuous operator and satisfies $L(K) \subset K$. Further, with the above assumptions, if we proceed in the lines of Lemma 2.5 in [16], we can find the existence of a eigenvalue and its corresponding eigenfunction ϕ. However, we prove the lemma here for our completeness.

Lemma 3.1. *Suppose that $\int_{1/4}^{3/4} s^4 q(s)b(s)\,ds > 0$. Let $r(L)$ be the spectral radius of L. Then $r(L) > 0$ and there exists $\phi \in K\backslash\{0\}$ such that $L\phi = r(L)\phi$.*

Proof. For $u \in K$, $t \in [1/4, 3/4]$, and using (2.7) we have

$$\begin{aligned} Lu(t) &= \int_0^1 G(t,s)b(s)q(s)u(s)\,ds \\ &\geq \int_{1/4}^{3/4} G(t,s)b(s)q(s)u(s)\,ds \\ &\geq \frac{\|u\|}{24} \int_{1/4}^{3/4} G(t,s)b(s)q(s)\,ds. \end{aligned}$$

Then

$$\begin{aligned} L^2u(t) &\geq \int_0^1 G(t,s_1)b(s_1)q(s_1) \left(\frac{\|u\|}{24} \int_{1/4}^{3/4} G(s_1,s_2)b(s_2)q(s_2)\,ds_2 \right) ds_1 \\ &\geq \frac{\|u\|}{24} \int_{1/4}^{3/4} \int_{1/4}^{3/4} G(t,s_1)G(s_1,s_2)b(s_1)b(s_2)q(s_1)q(s_2)\,ds_2 ds_1. \end{aligned}$$

By induction, we have

$$\begin{aligned}
\|L^n u\| \geq & \frac{\|u\|}{24} \max_{0\leq t\leq 1} \underbrace{\int_{1/4}^{3/4}\int_{1/4}^{3/4}\cdots\int_{1/4}^{3/4}}_{n-times} G(t,s_n)G(s_n,s_{n-1}),\cdots G(s_2,s_1) \\
& \qquad q(s_n)q(s_{n-1})\cdots q(s_1)b(s_n)b(s_{n-1})\cdots b(s_1)\, ds_1 ds_2\cdots ds_n \\
\geq & \frac{\|u\|}{24} \max_{0\leq t\leq 1} \underbrace{\int_{1/4}^{3/4}\int_{1/4}^{3/4}\cdots\int_{1/4}^{3/4}}_{n-times} \left(\frac{2t^2}{3}\right)\left(\frac{s_n^2}{2}\right)\left(\frac{2s_n^2}{3}\right)\left(\frac{s_{n-1}^2}{2}\right)\cdots\left(\frac{2s_2^2}{3}\right)\left(\frac{s_1^2}{2}\right) \\
& \qquad q(s_n)q(s_{n-1})\cdots q(s_1)b(s_n)b(s_{n-1})\cdots b(s_1)\, ds_1 ds_2\cdots ds_n \\
= & \frac{\|u\|}{24} \max_{0\leq t\leq 1} \left(\frac{2t^2}{3}\right) \underbrace{\int_{1/4}^{3/4}\int_{1/4}^{3/4}\cdots\int_{1/4}^{3/4}}_{n-times} \left(\frac{s_n^2}{2}\right)\left(\frac{2s_n^2}{3}\right)\left(\frac{s_{n-1}^2}{2}\right)\cdots\left(\frac{2s_2^2}{3}\right)\left(\frac{s_1^2}{2}\right) \\
& \qquad q(s_n)q(s_{n-1})\cdots q(s_1)b(s_n)b(s_{n-1})\cdots b(s_1)\, ds_1 ds_2\cdots ds_n \\
= & \frac{\|u\|}{24} \max_{0\leq t\leq 1} \left(\frac{2t^2}{3}\right) \left(\int_{1/4}^{3/4}\frac{s^2}{2}q(s)b(s)\,ds\right)\left(\int_{1/4}^{3/4}\frac{2s^2}{3}\frac{s^2}{2}q(s)b(s)\,ds\right)^{n-1} \\
= & \frac{\|u\|}{24}\frac{2}{3}\left(\int_{1/4}^{3/4}\frac{s^2}{2}q(s)b(s)\,ds\right)\frac{1}{3^{n-1}}\left(\int_{1/4}^{3/4}s^4 q(s)b(s)\,ds\right)^{n-1}.
\end{aligned}$$

Consequently,

$$\|L^n\|\|u\| \geq \|L^n u\| \geq \frac{\|u\|}{24}\frac{2}{3}\left(\int_{1/4}^{3/4}\frac{s^2}{2}q(s)b(s)\,ds\right)\frac{1}{3^{n-1}}\left(\int_{1/4}^{3/4}s^4 q(s)b(s)\,ds\right)^{n-1},$$

which implies that

$$\|L^n\| \geq \frac{1}{24}\frac{1}{3^n}\left(\int_{1/4}^{3/4}s^2 q(s)b(s)\,ds\right)\left(\int_{1/4}^{3/4}s^4 q(s)b(s)\,ds\right)^{n-1}.$$

By Gelfand's formula for spectral radius, we have

$$\begin{aligned}
r(L) &= \lim_{n\to\infty}(\|L^n\|)^{1/n} \\
&\geq \lim_{n\to\infty}\left[\frac{1}{12}\frac{1}{3^n}\left(\int_{1/4}^{3/4}\frac{s^2}{2}q(s)b(s)\,ds\right)\left(\int_{1/4}^{3/4}s^4 q(s)b(s)\,ds\right)^{n-1}\right]^{1/n} \\
&= \frac{1}{3}\int_{1/4}^{3/4}s^4 q(s)b(s)\,ds > 0.
\end{aligned}$$

Now the rest of the proof follows from Krein-Rutman theorem. □

As discussed in Section 1, the main theorems of this section uses the eigenvalues and their corresponding eigenfunctions to provide inequalities on the function $f(t,u)$ and ranges on λ for the existence of positive solutions to (1.1). By Lemma 3.1, there exists a positive eigenvalue, say $\lambda_{1,qb} := \frac{1}{r(L)}$ of (1.1), which we call first eigenvalue. Let us denote $\phi_{1,qb}$ be the associated eigenfunction. Then by Lemma 3.1, $\lambda_{1,qb}$ and $\phi_{1,qb}$ satisfy the properties $\phi_{1,qb} > 0$ and $\lambda_{1,qb} L\phi_{1,qb} = \phi_{1,qb}$, that is, $L\phi_{1,qb} = r(L)\phi_{1,qb}$.

Theorem 3.2. *Assume that there exist a continuous function $b : [0,1] \to (0,\infty)$ and positive constants c, δ, and R with $c > 1$ and $0 < \delta < R$ such that*

(H1) $f(t,u) \leq b(t)u$ *for* $u \in (0,\delta)$ *uniformly for* $t \in (0,1)$

and

(H2) $f(t,u) \geq cb(t)u$ *for* $u \geq R$ *uniformly for* $t \in (0,1)$.

Then the BVP (1.1) *has a positive solution for every λ with*

$$\frac{\lambda_{1,qb}}{c} < \lambda < \lambda_{1,qb}.$$

Proof. To apply Theorem 2.2, let $r \in (0,\delta)$. We claim that the integral equation

$$u(t) = \mu Tu, \;\; 0 < \mu < 1, \tag{3.4}$$

has no solution with norm r. If $u_0(t)$ was such a solution, then $u_0(t)$ is a solution of

$$u_0(t) = \mu\lambda \int_0^1 G(t,s)q(s)f(s,u_0(s))\,ds. \tag{3.5}$$

Multiplying bothsides of (3.5) by $b(t)q(t)\phi_{1,qb}(t)$ and integrating from 0 to 1, we obtain

$$\begin{aligned}\int_0^1 b(t)q(t)\phi_{1,qb}(t)u(t)\,dt =&\mu\lambda \int_0^1 b(t)q(t)\phi_{1,qb}(t)\left(\int_0^1 G(t,s)q(s)f(s,u_0(s))\,ds\right)dt\\ \leq&\mu\lambda \int_0^1 b(t)q(t)\phi_{1,qb}(t)\left(\int_0^1 G(t,s)q(s)b(s)u_0(s)\,ds\right)dt\\ =&\mu\lambda \int_0^1 q(s)b(s)u_0(s)\left(\int_0^1 G(s,t)b(t)q(t)\phi_{1,qb}(t)\,dt\right)ds\\ =&\mu\lambda\frac{1}{\lambda_{1,qb}}\int_0^1 q(s)b(s)u_0(s)\phi_{1,qb}(s)\,ds\\ <&\mu\int_0^1 q(s)b(s)u_0(s)\phi_{1,qb}(s)\,ds,\end{aligned}$$

which is a contradiction since $\mu < 1$. Thus, our claim holds, that is, (3.4) has no solution with norm r, and so condition (b) of Theorem 2.2 is satisfied.

On the set

$$D = \{u \in K : r \leq \|u\| \leq R\}$$

it is again clear that $T : D \to K$ is compact and continuous. To prove that condition (a) of Theorem 2.2 is satisfied we need to show that for any $\bar{R} \geq R$, the problem $u = \mu Tu$, $\mu > 1$, has no solution of norm $\|\bar{R}\|$. If this was not the case, then there is a sequence $\{R_n\}_{n=1}^{\infty} \to \infty$ as $n \to \infty$, with $R_n \geq \bar{R}$, a sequence $\{\mu_n\}_{n=1}^{\infty}$ with $\mu_n > 1$, and a sequence of functions $\{u_n\}_{n=1}^{\infty}$ with $\|u_n\| = R_n$ such that $u_n = \mu_n T u_n$ holds, that is,

$$u_n(t) = \mu_n \lambda \int_0^1 G(t,s)q(s)f(s,u_n(s))\,ds,\ 0 < t_n < 1,$$

for $n = 1, 2, \ldots,$ which further by using (H2), gives

$$u_n(t) \geq c\mu_n \lambda \int_0^1 G(t,s)q(s)b(s)u_n(s)\,ds,\ 0 < t_n < 1, \tag{3.6}$$

Multiplying both sides of (3.12) by $b(t)q(t)\phi_{1,qb}(t)$ and integrating from 0 to 1, we obtain

$$\begin{aligned}\int_0^1 b(t)q(t)\phi_{1,qb}(t)u(t)\,dt &\geq c\mu_n\lambda \int_0^1 b(t)q(t)\phi_{1,qb}(t)\left(\int_0^1 G(t,s)q(s)b(s)u_n(s)\,ds\right)dt\\ &= c\mu_n\lambda \int_0^1 q(s)b(s)u_n(s)\left(\int_0^1 G(s,t)b(t)q(t)\phi_{1,qb}(t)\,dt\right)ds\\ &= c\mu_n\lambda \frac{1}{\lambda_{1,qb}}\int_0^1 q(s)b(s)u_n(s)\phi_{1,qb}(s)\,ds\\ &> c\mu_n \int_0^1 q(s)b(s)u_n(s)\phi_{1,qb}(s)\,ds,\end{aligned}$$

which is a contradiction, because $\mu_n > 1$. Hence, condition (a) of Theorem 2.2 is satisfied. The proof of condition (c) of Theorem 2.2 follows from Lemma 2.4. By Theorem 2.2, the BVP (1.1) has a positive solution in D, and this proves the theorem. □

Theorem 3.3. *Assume that there exists a continuous function $b : [0,1] \to (0,\infty)$ and positive constants c, δ, and R with $c > 1$ and $0 < \delta < R$ such that*

(H3) $f(t,u) \leq b(t)u$ *for* $u \geq R$ *uniformly for* $t \in (0,1)$

and

(H4) $f(t,u) \geq cb(t)u$ *for* $u \in (0,\delta)$ *uniformly for* $t \in (0,1)$.

*Then the BVP (*1.1*) has a positive solution for every λ with*

$$\frac{\lambda_{1,qb}}{c} < \lambda < \lambda_{1,qb}.$$

Proof. We shall use Theorem 2.1 to prove the theorem. Let $r \in (0,\delta)$ and $u(t)$ be a solution of $u = \mu T u$ with $\mu > 1$. We claim that $\|u\| \neq r$. If this is not true, there exists a solution $u_0(t)$ of $u(t) = \mu T u(t),\ \mu > 1$, and $u_0(t)$ satisfies the property $\|u_0\| = r$. Then $u_0(t)$ is a solution of

$$u_0(t) = \mu\lambda \int_0^1 G(t,s)q(s)f(s,u_0(s))\,ds.\ 0 < t < 1,\ \mu > 1. \tag{3.7}$$

Multiplying both sides of Eq.(3.13) by $b(t)q(t)\phi_{1,qb}(t)$, integrating from 0 to 1, and using (H4) and the fact that $\lambda > \frac{\lambda_{1,qb}}{c}$, we obtain

$$\int_0^1 b(t)q(t)\phi_{1,qb}(t)u_0(t)\,dt > \mu \int_0^1 b(t)q(t)\phi_{1,qb}(t)u_0(t)\,dt,$$

which is a contradiction because $\mu > 1$. So our claim holds. Thus, if we consider the set

$$D = \{u \in K : r \leq \|u\| \leq R\},$$

then $T : D \to K$ is compact and continuous. Furthermore, for the above choice of r, condition (b) of Theorem 2.1 is satisfied.

Next, we show condition (a) of Theorem 2.1 is satisfied. Let $u(t) \in D$ be a solution of $u = \mu Tu$ with $\mu < 1$. We shall show that $\|u\| \neq R$. It suffices to show that the problem $u = \mu Tu$, $\mu < 1$, has no solution of norm $\bar{R}$ for any $\bar{R} \geq R$. Suppose there is such a solution $u_1(t)$ with $\|u_1\| = R_0 \geq R$. Then proceeding as before, using $\lambda < \lambda_{1,qb}$, we can obtain the contradiction

$$\int_0^1 q(t)b(t)\phi_{1,qb}(t)u_1(t)dt < \mu \int_0^1 q(t)\phi_{1,qb}(t)b(t)u_1(t)dt.$$

Hence, our claim is true and condition (a) of Theorem 2.1 holds. The proof that condition (c) of Theorem 2.1 holds is similar to the proof of Lemma 2.4. Therefore, by Theorem 2.1, the BVP (1.1) has at least one positive solution $u(t)$. This completes the proof of the theorem. □

Example 3.4. Consider the problem

$$\begin{cases} u''''(t) = & \lambda u^3, \quad t \in [0,1], \\ u(0) = & u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{3.8}$$

Here $f(t,u) = u^3$ and $q(t) = 1$. Setting $b(t) = 1$, it is easy to verify that $\lambda_{1,qb} = 16\pi^4$ is the first eigenvalue of

$$\begin{cases} u''''(t) = & \lambda u, \quad t \in [0,1], \\ u(0) = & u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

Let $c > 1$ be a constant and choose $\delta \in (0,1)$ such that $c\delta^2 > 1$. Then for $u \in (0,\delta)$, we have $f(t,u) = u^3 < u$. Hence, condition $(H1)$ of Theorem 3.2 is satisfied. Set $R = c\delta$; then for $u \geq R$, we have $f(t,u) = u^3 = u^2 u > c^2\delta^2 u = c \cdot c\delta^2 u > cu$. Thus, condition $(H2)$ of Theorem 3.2 is satisfied. Hence, by Theorem 3.2, (3.8) has a positive solution for every $\frac{16\pi^4}{c} < \lambda < 16\pi^4$.

Example 3.5. Consider two constants $\delta \in (0,1)$ and $R > 1$ and choose $c = \frac{1}{\delta^2}$. Then by Theorem 3.3, the problem

$$\begin{cases} u''''(t) = & \lambda u^{-1}, \quad t \in [0,1], \\ u(0) = & u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

has a positive solution $u(t)$ for each $\lambda \in (16\delta^2\pi^4, 16\pi^4)$.

Finally, we consider the case, when $\lambda = 1$. Then the BVP (1.1) becomes

$$\begin{cases} u''''(t) & = q(t)f(t,u(t)), \quad t \in [0,1], \\ u(0) = & u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{3.9}$$

where $f \in \mathcal{C}([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$ and $q \in \mathcal{C}([0,1], \mathbb{R}_+)$.

Proceeding as in the lines of Theorems 3.2 and 3.3, we can obtain the following results.

Theorem 3.6. *Assume that there exist a continuous function $b : [0,1] \to (0,\infty)$ and positive constants c, δ, and R with $c \geq 1$ and $0 < \delta < R$ such that*

(H5) $f(t,u) \leq \lambda_{1,qb} b(t) u$ *for $u \in (0,\delta)$ uniformly for $t \in (0,1)$*

and

(H6) $f(t,u) \geq c\lambda_{1,qb} b(t) u$ *for $u \geq R$ uniformly for $t \in (0,1)$.*

Then the BVP (3.9) has a positive solution.

Proof. To apply Theorem 2.2, let $r \in (0,\delta)$. We claim that the integral equation

$$u(t) = \mu T u, \ 0 < \mu < 1, \tag{3.10}$$

has no solution with norm r. If $u_0(t)$ was such a solution, then $u_0(t)$ is a solution of

$$u_0(t) = \mu \int_0^1 G(t,s) q(s) f(s, u_0(s))\, ds. \tag{3.11}$$

Multiplying bothsides of (3.11) by $b(t)q(t)\phi_{1,qb}(t)$ and integrating from 0 to 1, we obtain

$$\begin{aligned}
\int_0^1 b(t)q(t)\phi_{1,qb}(t)u(t)\,dt =&\mu \int_0^1 b(t)q(t)\phi_{1,qb}(t) \left(\int_0^1 G(t,s)q(s)f(s,u_0(s))\,ds\right) dt \\
\leq&\mu\lambda_{1,qb} \int_0^1 b(t)q(t)\phi_{1,qb}(t) \left(\int_0^1 G(t,s)q(s)b(s)u_0(s)\,ds\right) dt \\
=&\mu\lambda_{1,qb} \int_0^1 q(s)b(s)u_0(s) \left(\int_0^1 G(s,t)b(t)q(t)\phi_{1,qb}(t)\,dt\right) ds \\
=&\mu\lambda_{1,qb} \frac{1}{\lambda_{1,qb}} \int_0^1 q(s)b(s)u_0(s)\phi_{1,qb}(s)\,ds \\
<&\mu \int_0^1 q(s)b(s)u_0(s)\phi_{1,qb}(s)\,ds,
\end{aligned}$$

which is a contradiction since $\mu < 1$. Thus, our claim holds, that is, (3.10) has no solution with norm r, and so condition (b) of Theorem 2.2 is satisfied.

On the set

$$D = \{u \in K : r \leq \|u\| \leq R\}$$

it is again clear that $T : D \to K$ is compact and continuous. To prove that condition (a) of Theorem 2.2 is satisfied we need to show that for any $\bar{R} \geq R$, the problem $u = \mu T u$, $\mu > 1$, has no solution of norm $\|\bar{R}\|$. If this was not the case, then there is a sequence $\{R_n\}_{n=1}^{\infty} \to \infty$ as $n \to \infty$, with $R_n \geq \bar{R}$, a sequence $\{\mu_n\}_{n=1}^{\infty}$ with $\mu_n > 1$, and a sequence of functions $\{u_n\}_{n=1}^{\infty}$ with $\|u_n\| = R_n$ such that $u_n = \mu_n T u_n$ holds, that is,

$$u_n(t) = \mu_n \int_0^1 G(t,s)q(s)f(s,u_n(s))\,ds, \ 0 < t_n < 1,$$

for $n = 1, 2, \ldots,$ which further by using (H6), gives

$$u_n(t) \geq c\mu_n \lambda_{1,qb} \int_0^1 G(t,s)q(s)b(s)u_n(s)\,ds,\ 0 < t_n < 1, \tag{3.12}$$

Multiplying both sides of (3.12) by $b(t)q(t)\phi_{1,qb}(t)$ and integrating from 0 to 1, we obtain

$$\begin{aligned}\int_0^1 b(t)q(t)\phi_{1,qb}(t)u(t)\,dt &\geq c\mu_n \int_0^1 b(t)q(t)\phi_{1,qb}(t)\left(\int_0^1 G(t,s)q(s)b(s)u_n(s)\,ds\right)dt\\ &= c\mu_n \int_0^1 q(s)b(s)u_n(s)\left(\int_0^1 G(s,t)b(t)q(t)\phi_{1,qb}(t)\,dt\right)ds\\ &= c\mu_n \frac{1}{\lambda_{1,qb}} \int_0^1 q(s)b(s)u_n(s)\phi_{1,qb}(s)\,ds\\ &> c\mu_n \int_0^1 q(s)b(s)u_n(s)\phi_{1,qb}(s)\,ds,\end{aligned}$$

which is a contradiction, because $\mu_n > 1$. Hence, condition (a) of Theorem 2.2 is satisfied. The proof of condition (c) of Theorem 2.2 follows from Lemma 2.4. By Theorem 2.2, the BVP (3.9) has a positive solution in D, and this proves the theorem. □

Theorem 3.7. *Assume that there exists a continuous function $b : [0,1] \to (0,\infty)$ and positive constants c, δ, and R with $c \geq 1$ and $0 < \delta < R$ such that*

(H7) $f(t,u) \leq \lambda_{1,qb}b(t)u$ *for* $u \geq R$ *uniformly for* $t \in (0,1)$

and

(H8) $f(t,u) \geq c\lambda_{1,qb}b(t)u$ *for* $u \in (0,\delta)$ *uniformly for* $t \in (0,1)$.

*Then the BVP (*1.1*) has a positive solution.*

Proof. We shall use Theorem 2.1 to prove the theorem. Let $r \in (0,\delta)$ and $u(t)$ be a solution of $u = \mu Tu$ with $\mu > 1$. We claim that $\|u\| \neq r$. If this is not true, there exists a solution $u_0(t)$ of $u(t) = \mu Tu(t),\ \mu > 1$, and $u_0(t)$ satisfies the property $\|u_0\| = r$. Then $u_0(t)$ is a solution of

$$u_0(t) = \mu \int_0^1 G(t,s)q(s)f(s,u_0(s))\,ds.\ 0 < t < 1,\ \mu > 1. \tag{3.13}$$

Multiplying both sides of Eq.(3.13) by $b(t)q(t)\phi_{1,qb}(t)$, integrating from 0 to 1, and using (H8), we obtain

$$\int_0^1 b(t)q(t)\phi_{1,qb}(t)u_0(t)\,dt > \mu \int_0^1 b(t)q(t)\phi_{1,qb}(t)u_0(t)\,dt,$$

which is a contradiction because $\mu > 1$. So our claim holds. Thus, if we consider the set

$$D = \{u \in K : r \leq \|u\| \leq R\},$$

then $T : D \to K$ is compact and continuous. Furthermore, for the above choice of r, condition (b) of Theorem 2.1 is satisfied.

Next, we show condition (a) of Theorem 2.1 is satisfied. Let $u(t) \in D$ be a solution of $u = \mu Tu$ with $\mu < 1$. We shall show that $\|u\| \neq R$. It suffices to show that the problem $u = \mu Tu$, $\mu < 1$, has no solution of norm $\bar{R}$ for any $\bar{R} \geq R$. Suppose there is such a solution $u_1(t)$ with $\|u_1\| = R_0 \geq R$. Then proceeding as before, we can obtain the contradiction

$$\int_0^1 q(t)b(t)\phi_{1,qb}(t)u_1(t)dt < \mu \int_0^1 q(t)\phi_{1,qb}(t)b(t)u_1(t)dt.$$

Hence, our claim is true and condition (a) of Theorem 2.1 holds. The proof that condition (c) of Theorem 2.1 holds is similar to the proof of Lemma 2.4. Therefore, by Theorem 2.1, the BVP (1.1) has at least one positive solution $u(t)$. This completes the proof of the theorem.

□

Remark 3.8. The conditions (H5) and (H6) of Theorem 3.6, and (H7) and (H8) of Theorem 3.7 are implied by conditions the following theorems.

Theorem 3.9. *Assume that there exists a continuous function $b : [0,1] \to (0,\infty)$ and a constant $c > 1$ such that*

$$\limsup_{u \to 0+} \frac{f(t,u)}{u} \leq b(t)\lambda_{1,q} \text{ and } \liminf_{u \to \infty} \frac{f(t,u)}{u} > cb(t)\lambda_{1,q} \tag{3.14}$$

holds uniformly for $t \in (0,1)$. Then the BVP (1.1) has a positive solution.

Theorem 3.10. *Assume that there exists a continuous function $b : [0,1] \to (0,\infty)$ and a constant $c > 1$ such that*

$$\limsup_{u \to \infty} \frac{f(t,u)}{u} \leq b(t)\lambda_{1,q} \text{ and } \liminf_{u \to 0+} \frac{f(t,u)}{u} > cb(t)\lambda_{1,q} \tag{3.15}$$

holds uniformly for $t \in (0,1)$. Then the BVP (1.1) has a positive solution.

Remark 3.11. Since the existence of a positive solution of (1.1) is equivalent to the existence of a positive solution of the integral equation (1.2), where $G(t,s)$ is the Green's function defined in (2.4), then our Theorems 3.9–3.10 can also be applied to (1.2). Webb and Lan [16] proved some existence theorems for the general integral equation of the form (1.2) with general Green's function $G(t,s)$ which is measurable and satisfies the properties (3.3), there exists a subinterval $[a,b] \subseteq [0,1]$, a function $\Phi \in L^\infty[0,1]$, and a constant $\mu \in (0,1]$ such that

$$q\Phi \in L^1[0,1],\ q \geq 0 \text{ a.e. and } \int_a^b q(s)\Phi(s)\,ds > 0,$$

$$G(t,s) \leq \Phi(s) \text{ for } t \in [0,1] \text{ and a.e.} s \in [0,1],$$

and

$$G(t,s) \geq \mu\Phi(s) \text{ for } t \in [a,b] \text{ and a.e.} s \in [0,1]$$

and f satisfies Caratheodory conditions. In Section 2, we have proved the above conditions on $G(t,s)$ with $\mu = 1/24$ and $\Phi(s) = g(s)$. When $b(t) \equiv 1$ and $c = 1$, then the conditions of Theorems 3.9-3.10 are equivalent to the conditions of Theorem 4.1 due to Webb and Lan [16].

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