Positive Solutions of Riemann-Liouville Fractional
Differential Equations with
Nonlocal Boundary Conditions

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Abstract

We study the existence of at least two positive solutions for a system of Riemann-Liouville fractional differential equations with multipoint boundary conditions. We use method of iterations to prove our results. We require the nondecreasing property of a nonlinear function on a certain range to prove our results.

1 Introduction

In this paper, we use monotone iteration method to obtain sufficient conditions for the existence of at least one positive solution of the scalar equation

$$D_{0+}^{\alpha} u(t) + a(t) f(t, u(t)) = 0,\ t \in (0, 1) \tag{1.1}$$

with the multipoint fractional BCs

$$u(0) = u'(0) = ... = u^{(n-2)} = 0,\ \ D_{0+}^{p} u(t)|_{t=1} = \sum_{i=1}^{m} a_i D_{0+}^{q} u(t)|_{t=\xi_i}, \tag{1.2}$$

where $\alpha \in \mathbb{R}, \alpha \in (n-1, n], n \in \mathbb{N}, n \geq 3, \xi_i \in \mathbb{R}$ for all $i = 1, 2, ..., m(m \in \mathbb{N}), 0 < \xi_1 < \xi_2 < .... < \xi_m < 1, p, q, \in \mathbb{R}, p \in [1, n-2], q \in [0, p]$. Further, $D_{0+}^{\alpha}, D_{0+}^{p}$ and D_{0+}^{q} denote the Riemann-Liouville derivative of order α, p and q respectively, and the nonlinearity on f may change sign and may be singular at $t = 0$ or $t = 1$, $\mathbb{N}$ the set of natural numbers and $\mathbb{R}$ denote the set of real numbers with $\mathbb{R}_+ = [0, \infty)$.

In a recent work, Henderson and Luca [4] used Guo-Krasnosel'skii fixed point theorem to find sufficient conditions for the existence of positive solutions of (1.1). In [5] and [6], Handerson and Luca used fixed point index approach to study the existence and multiplicity of positive solutions to the coupled system of fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t) + a(t)f(t, u(t), v(t)) = 0, \, t \in (0, 1), \\ D_{0+}^{\beta}u(t) + b(t)g(t, u(t), v(t)) = 0, \, t \in (0, 1), \end{cases} \tag{1.3}$$

with the multipoint boundary conditions (BCs)

$$\begin{cases} u^{(j)}(0) = 0, \, j = 0, ..., n-2, \, D_{0+}^{p_1}u(t)|_{t=1} = \sum_{i=1}^{N} a_i D_{0+}^{q_1}u(t)|_{t=\xi_i}, \\ v^{(k)}(0) = 0, \, k = 0, ..., m-2, \, D_{0+}^{p_2}v(t)|_{t=1} = \sum_{i=1}^{M} b_i D_{0+}^{q_2}v(t)|_{t=\eta_i}, \end{cases} \tag{1.4}$$

where $\alpha, \beta \in \mathbb{R}, \, \alpha \in (n-1, n], \beta \in (m-1, m], m, n \in \mathbb{N}, \, m, n \geq 3, \, p_1, p_2, q_1, q_2 \in \mathbb{R}, p_1 \in [1, n-2], p_2 \in [1, m-2], q_1 \in [0, p_1], q_2 \in [0, p_2], \xi_i, a_i \in \mathbb{R}$ for $i = 1, 2, ..., N, N \in \mathbb{N}, 0 < \xi_1 < \xi_2 < ... < \xi_N \leq 1, \eta_i, b_i \in \mathbb{R}$ for $i = 1, 2, ..., M, M \in \mathbb{N}, 0 < \eta_1 < \eta_2 < ... < \eta_M \leq 1, D_{0+}^{\alpha}, D_{0+}^{\beta}, D_{0+}^{p_1}, D_{0+}^{q_1}, D_{0+}^{p_2}$ and $D_{0+}^{q_2}$ denote the Riemann-Liouville derivatives of orders $\alpha, \beta \, p_1, q_1, p_2$ and q_2 respectively and f and $g : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ are continuous functions, and $a, b : [0, 1] \to [0, \infty)$ are continuous functions.

The works in [6] extends the work in [5], where fixed point index method was used when f and g are nonsingular or singular at the points $t = 0$ and/or $t = 1$. Xie and Xie [14] used fixed-point index theory to obtain sufficient conditions for the existence and multiplicity of positive solutions of the system of higher-order nonlinear fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda_1 f_1(t, u(t), v(t)) = 0, \, t \in (0, 1), \\ D_{0+}^{\beta}u(t) + \lambda_2 f_2(t, u(t), v(t)) = 0, \, t \in (0, 1), \end{cases} \tag{1.5}$$

together with the BCs containing the fractional derivatives

$$\begin{cases} D_{0+}^{\mu}u(1) = \eta_1 D_{0+}^{\mu}u(\xi_1), \\ D_{0+}^{\gamma}v(1) = \eta_2 D_{0+}^{\gamma}v(\xi_2). \end{cases} \tag{1.6}$$

where $\lambda_1 > 0, \lambda_2 > 0$ are parameters $D_{0+}^{\alpha}, D_{0+}^{\beta}, D_{0+}^{\mu}$ and D_{0+}^{γ} are the standard Riemman-Liouville fractional derivatives of orders α, β, μ and γ respectively, with $\alpha, \beta \in (n-1, n], n \geq 3, 1 \leq \mu, \gamma \leq n-2, n \in \mathbb{N}, \xi_1, \xi_2 \in (0, 1), 0 < \eta_1 \xi_1^{\alpha-\mu-1} < 1, \, 0 < \eta_2 \xi_2^{\beta-\gamma-1} < 1$, and $f_i \in C([0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+), i = 1, 2$. The assumption on the functions f and g in (1.3) are more general than the ones in [6] and [14], that is, the assumption of the function f

in (1.1), and the functions f_1 and f_2 in (1.5). It is assumed in [6] that the function f and $g : [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous. Luca [7] used Guo-Krasnoselskii's fixed point theorem and extended the works of Henderson and Luca [5] to a system of three fractional differential equations with nonlocal boundary conditions containing fractional derivatives. Recently, Padhi et al [11, 12] used Leray Schauder alternate, Avery-peterson fixed point theorem and Fixed point index approach tp study the existence of positive solutions of (1.3) together with the boundary condition (1.4). The results of Padhi et al. [11, 12] extends the works of [5, 6].

Padhi and Pati [8] used fixed point index approach, and Padhi et al. [10] used Krasnosel'skii fixed point theorem and a fixed point theorem due to Avery and Peterson to study the existence and multiplicity of positive solutions of equations of the form (1.1) with Riemann-Stieltjes type integral boundary conditions without the parameter λ.

By a positive solution of the problem (1.1)–(1.2), we mean a function $u \in (C([0,1] : \mathbb{R}_+)$ satisfying (1.1)–(1.2) with $u(t) > 0$ for all $t \in (0,1]$.

The motivation for the present work has come from a recent work due to the Padhi and Prasad [9]. In this work, we shall use monotone iteration theorem to obtain sufficient conditions for the existence of positive solutions of (1.1). One can observe from the assumption (3.4) (See Theorem 3.2) that, we do not require any super-linearity or sub-linearity on f either at 0 or ∞. The only assumption we require on f is that f must be monotonically nondecreasing with respect to u in some subinterval, say $[0, R]$ of $[0, \infty)$. The function f may decrease or non-decrease or indentically zero on the other half of the interval $[0, R]$. This shows that our assumption (3.4) is not comparable with the results in [4, 11, 12].

This work has been divided into three sections. Section 2 contains the basic results and the well-known monotone iterative method (see [1, 2, 3] or Theorem 7.A in [4]). Section 3 contains the main result of this paper. The study is supplemented with examples to illustrate the applicability of our results.

2   Preliminaries

Let us consider the fractional differential equation

$$D_{0+}^{\alpha} u(t) + x(t) = 0, \ t \in (0,1) \tag{2.1}$$

with the multipoint BCs

$$u^{(j)}(0) = 0, \ j = 0, 1, \cdots, n-2, \ \ D_{0+}^{p} u(t)|_{t=1} = \sum_{i=1}^{N} a_i D_{0+}^{q} u(t)|_{t=\xi_i}, \tag{2.2}$$

where $\alpha \in (n-1, n], n \in \mathbb{N}, n \geq 3, a_i, \xi_i \in \mathbb{R}, i = 1, 2, \cdots, N(N \in \mathbb{N}), 0 < \xi_1 < \xi_2 < \cdots < \xi_N \leq 1, \ p, q \in \mathbb{R}, p \in [1, n-2], \ q \in [0, p]$, and $x \in C[0,1]$.

If we set

$$\Delta := \frac{\Gamma(\alpha)}{\Gamma(\alpha - p)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha - q)} \sum_{i=1}^{N} a_i \xi_i^{\alpha - q - 1} \neq 0,$$

then a unique solution of (2.1)–(2.2) is given by the integral equation

$$u(t) = \int_0^1 G(t, s) x(s) \, ds, \ t \in [0, 1],$$

where

$$G(t, s) = g_1(t, s) + \frac{t^{\alpha - 1}}{\Delta_1} \sum_{i=1}^{N} a_i g_2(\xi_i, s), \ (t, s) \in [0, 1] \times [0, 1], \qquad (2.3)$$

and

$$g_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha - 1}(1 - s)^{\alpha - p - 1} - (t - s)^{\alpha - 1}, & 0 \le s \le t \le 1, \\ t^{\alpha - 1}(1 - s)^{\alpha - p - 1}, & 0 \le t \le s \le 1, \end{cases}$$

$$g_2(t, s) = \frac{1}{\Gamma(\alpha - q)} \begin{cases} t^{\alpha - q - 1}(1 - s)^{\alpha - p - 1} - (t - s)^{\alpha - q - 1}, & 0 \le s \le t \le 1, \\ t^{\alpha - q - 1}(1 - s)^{\alpha - p - 1}, & 0 \le t \le s \le 1. \end{cases}$$

The functions $g_1(t, s)$ and $g_2(t, s)$ have the following properties.

$(A1)$ $g_1(t, s) \le h_1(s)$ for all $t, s \in [0, 1]$, where

$$h_1(s) = \frac{1}{\Gamma(\alpha)} (1 - s)^{\alpha - p - 1}(1 - (1 - s)^p), \ s \in [0, 1];$$

$(A2)$ $g_1(t, s) \ge t^{\alpha - 1} h_1(s)$ for all $t, s \in [0, 1]$;

$(A3)$ $g_1(t, s) \le \frac{t^{\alpha - 1}}{\Gamma(\alpha)}$ for all $t, s \in [0, 1]$;

$(A4)$ $g_2(t, s) \ge t^{\alpha - q - 1} h_2(s)$ for all $t, s \in [0, 1]$, where

$$h_2(s) = \frac{1}{\Gamma(\alpha - q)} (1 - s)^{\alpha - p - 1}(1 - (1 - s)^{p - q}), \ s \in [0, 1];$$

$(A5)$ $g_2(t, s) \le \frac{t^{\alpha - q - 1}}{\Gamma(\alpha - q)}$ for all $t, s \in [0, 1]$;

$(A6)$ the functions g_1 and g_2 are continuous on $[0, 1] \times [0, 1]$, $g_1(t, s) \ge 0, g_2(t, s) \ge 0$ for all $t, s \in [0, 1]$ and $g_1(t, s) > 0, g_2(t, s) > 0$ for all $t, s \in (0, 1]$.

Let $\Delta > 0$ and $a_i \geq 0$ for all $i = 1, 2, \cdots, N$. An application of the properties $(A1) - (A6)$ on $g_1(t, s)$ and $g_2(t, s)$, we obtain the following inequalities on the Green's function $G(t, s)$.

$(A7)$ $G(t, s) \leq J(s)$ for all $t, s \in [0, 1]$, where

$$J(s) = h_1(s) + \frac{1}{\Delta} \sum_{i=1}^{N} a_i g_2(\xi_i, s), \ s \in [0, 1];$$

$(A8)$ $G(t, s) \geq t^{\alpha-1} J(s)$ for all $t, s \in [0, 1]$.

Setting $\phi(t) = t^{\alpha-1}$, the inequalities in $(A7) - (A8)$ can be interpreted into the following Harnack type inequality

$$\phi(t)J(s) \leq G(t, s) \leq J(s), \ t, s \in [0, 1], i = 1, 2. \tag{2.4}$$

Since the functions $g_i(t, s)$, $i = 1, 2, 3, 4$ are continuous on $t, s \in [0, 1]$, $g_i(t, s) \geq 0$ for $(t, s) \in [0, 1] \times [0, 1]$ with $g_i(t, s) > 0$ for $(t, s) \in (0, 1] \times (0, 1]$, we have (2.4) is valid for arbitrary $[a, b] \subset [0, 1]$, for which the inequality

$$\mu J(s) \leq G(t, s) \leq J(s), \ s \in [0, 1] \tag{2.5}$$

is a replacement for the inequality (2.4), where

$$\mu = \min_{t \in [a, b]} \phi(t) = c^{\alpha-1}$$

with

$$c = \min\{a, 1 - b\}. \tag{2.6}$$

It is well known that $[a, b] = [1/4, 3/4]$ is the optimal subinterval of $[0, 1]$ to work in a cone to obtain multiplicity results. In particular, for $[a, b] = [1/4, 3/4]$, we have $\mu = \frac{1}{4^{\alpha-1}}$.

Now, we provide some basic concepts of the cones in a Banach space and monotone iteration method.

Definition 2.1 *Let X be a real Banach space. A nonempty convex closed set $K \subset X$ is said to be a cone provided that*

(i) $kx \in K$ for all $x \in K$ and for all $k \geq 0$, and

(ii) $x, -x \in K$ implies $x = 0$.

We note that an operator is called completely continuous if it is continuous and maps bounded sets into precompact sets. We shall use the following well-known monotone iterative method (see [1, 2, 3] or Theorem 7.A in [4]).

Theorem 2.2 *Let X be a real Banach space and K be a cone in X. Assume that there exist constants v_0 and w_0 with $v_0 \leq w_0$ and $[v_0, w_0] \subset X$ such that*

(i) $T : [v_0, w_0] \to X$ *is completely continuous;*

(ii) T *is a monotonic increasing operator on $[v_0, w_0]$;*

(iii) v_0 *is a lower solution of T, that is, $v_0 \leq Tv_0$;*

(iv) w_0 *is an upper solution of T, that is, $Tw_0 \leq w_0$.*

Then T has a fixed point and the iterative sequence $v_{n+1} = Tv_n$ and $w_{n+1} = Tw_n$, $n = 1, 2, 3, \ldots$ with

$$v_0 \leq v_1 \leq v_2 \leq \ldots \leq v_n \leq \ldots \leq w_n \leq w_{n-1} \leq \ldots \leq w_1 \leq w_0$$

converges to v and w respectively, which are the greatest and lowest fixed points of T in $[v_0, w_0]$.

3   Main Result

In this section, we consider X is a Banach space with the max norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. Consider an operator $T : X \to X$ by

$$T(u)(t) = \int_0^1 G(t, s)a(s)f(s, u(s))ds. \tag{3.1}$$

Define a cone P on X by

$$P = \left\{ u \in X : \min_{t \in [1/4, 3/4]} u(t) \geq \frac{1}{4^{\alpha-1}} \|u\| \right\}.$$

Then u is a solution of (1.1)–(1.2) if and only if u is a fixed point of T in X.

Theorem 3.1 *Let $u(t)$ be a solution of (1.1)–(1.2). Then $u(t)$ is nondecreasing.*

Proof. Clearly $u(t)$ is a solution of the problem (1.1)–(1.2) if and only if $u(t)$ is a solution of the integral equation

$$u(t) = \int_0^1 G_1(t, s)a(s)f(s, u(s))ds, \quad t \in [0, 1]. \tag{3.2}$$

Differentiating both sides of (3.2) with respect to t, we obtain

$$
\begin{aligned}
u'(t) = {} & \frac{(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-p-1} f(s,u(s))ds \\
& - \frac{(\alpha-1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} f(s,u(s))ds \\
& + \frac{(\alpha-1)t^{\alpha-2}}{\Delta\Gamma(\alpha-q)} \sum_{i=1}^m a_i \Big[\int_0^1 \xi_i^{\alpha-q-1}(1-s)^{\alpha-p-1} f(s,u(s))\,ds \\
& \qquad - \int_0^{\xi_i} (\xi_i - s)^{\alpha-q-1} f(s,u(s))\,ds \Big].
\end{aligned}
$$

(3.3)

Let
$$
I_1 = t^{\alpha-2}(1-s)^{\alpha-p-1} - (t-s)^{\alpha-2},
$$
and
$$
I_2 = \xi_i^{\alpha-q-1}(1-s)^{\alpha-p-1} - (\xi_i - s)^{\alpha-q-1}.
$$

Let $I_1 \leq 0$ for $0 \leq s \leq t \leq 1$. Then for $s \leq t \leq 1$ and $p \geq 1$, we the inequality

$$
t^{\alpha-2}(1-s)^{\alpha-p-1} \leq t^{\alpha-2}\left(1-\frac{s}{t}\right)^{\alpha-2} \leq t^{\alpha-2}(1-s)^{\alpha-2},
$$

which gives $1 \leq (1-s)^p < 1$, a contradiction. Now let $I_2 \leq 0$ for $0 \leq s \leq \xi_i$. Clearly $\alpha - q - 1 > 0$. Since $\left(1-\frac{s}{\xi_i}\right) < 1 - s$, then

$$
\xi_i^{\alpha-q-1}(1-s)^{\alpha-p-1} \leq (\xi_i - s)^{\alpha-q-1} = \xi_i^{\alpha-q-1}\left(1-\frac{s}{\xi_i}\right)^{\alpha-q-1}
$$

implies that

$$
(1-s)^{\alpha-p-1} < \left(1-\frac{s}{\xi_i}\right)^{\alpha-q-1} \leq (1-s)^{\alpha-q-1}.
$$

Thus we obtain $1 \leq (1-s)^{p-q} < 1$, which is a contradiction. So $I_1 \geq 0$ and $I_2 \geq 0$ yields that $u(t)$ is nondecreasing in $[0,1]$. The lemma is proved.

Now we shall prove the main result of this paper.

Theorem 3.2 *If there exists a positive constant R such that*

$$
0 \leq f(t,u) \leq f(t,v) \leq \frac{R}{\int_0^1 J(s)a(s)\,ds} \quad \text{for } 0 \leq u \leq v \leq R \text{ and } 0 \leq t \leq 1, \qquad (3.4)
$$

and

$$
f(t,0) \neq 0 \ \text{ for all } 0 \leq t \leq 1 \qquad (3.5)
$$

are satisfied, then the problem (1.1)–(1.2) has at least two nondecreasing positive solutions.

Proof. Let $u \in P$. Then

$$\|Tu\| \leq \int_0^1 J(s)a(s)f(s, u(s))\, ds$$

and

$$\min_{t \in [1/4, 3/4]} Tu(t) \geq \mu \int_0^1 J(s)a(s)f(s, u(s))\, ds \geq \mu\|Tu\|$$

implies that $T : P \to P$. Also, T is well-defined. Set $v_0 = 0$ and $w_0 = R$; then $v_0 \leq w_0$. Now we shall show that $T : [v_0, w_0] \to P$ is completely continuous.

Define an open bounded set U on the Banach space X as

$$U = \{u(t) : u(t) \in X, \|u\| \leq R, t \in [0, 1]\}.$$

The for $u \in U$, we have

$$\|Tu\| \leq \int_0^1 J(s)a(s)f(s, u(s))\, ds \leq \int_0^1 J(s)a(s)f(s, R)\, ds \leq R,$$

which implies that $\|Tu\| \leq R$. Hence $T(U) \subseteq U$.

Since G, f, and a are continuous functions, T is continuous. Now, we shall show that T is a completely continuous operator. For this, let us assume

$$f^M = \max_{0 \leq t \leq 1}\{|f(t, u(t))| : u \in U\}, \text{ and } a^M = \max_{0 \leq t \leq 1} a(t).$$

Let t and $\tau \in [0, 1]$ be such that $t < \tau$. Then for any $u \in U$, we have

$$|T(u)(t) - T(u)(\tau)|$$

$$\leq \int_0^1 |G(t, s) - G(\tau, s)| a(s)| f(s, u(s))|ds$$

$$\leq f^M a^M \int_0^1 |G(t, s) - G(\tau, s)|ds$$

$$= f^M a^M \left[\int_0^t |g_1(t, s) - g_1(\tau, s)|ds + \int_t^\tau |g_1(t, s) - g_1(\tau, s)|ds \right.$$

$$\left. + \int_\tau^1 |g_1(t, s) - g_1(\tau, s)|ds + \frac{[\tau^{\alpha-1} - t^{\alpha-1}]}{\Delta \Gamma(\alpha - q)} \sum_{i=1}^N a_i \xi_i^{\alpha-q-1} \right]$$

$$= f^M a^M \left[\frac{1}{\Gamma(\alpha)} \int_0^t |t^{\alpha-1}(1 - s)^{\alpha-p-1} - (t - s)^{\alpha-1} - \tau^{\alpha-1}(1 - s)^{\alpha-p-1} + (\tau - s)^{\alpha-1}|ds \right.$$

$$+ \frac{1}{\Gamma(\alpha)} \int_t^\tau |t^{\alpha-1}(1 - s)^{\alpha-p-1} - \tau^{\alpha-1}(1 - s)^{\alpha-p-1} + (\tau - s)^{\alpha-1}|ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_\tau^1 |t^{\alpha-1}(1 - s)^{\alpha-p-1} - \tau^{\alpha-1}(1 - s)^{\alpha-p-1}|ds$$

$$\left. + \frac{(\tau^{\alpha-1} - t^{\alpha-1})}{\Delta_1 \Gamma(\alpha - q)} \sum_{i=1}^N a_i \xi_i^{\alpha-q-1} \right]$$

$$\leq f^M a^M \left[\frac{1}{\Gamma(\alpha)} |t^{\alpha-1} - \tau^{\alpha-1}| \int_0^t (1 - s)^{\alpha-p-1}ds + \frac{1}{\Gamma(\alpha)} \int_0^t |(\tau - s)^{\alpha-1} - (t - s)^{\alpha-1}|ds \right.$$

$$+ \frac{|t^{\alpha-1} - \tau^{\alpha-1}|}{\Gamma(\alpha)} \int_t^\tau (1 - s)^{\alpha-p-1}ds + \frac{1}{\Gamma(\alpha)} \int_t^\tau (\tau - s)^{\alpha-1}ds$$

$$\left. + \frac{|t^{\alpha-1} - \tau^{\alpha-1}|}{\Gamma(\alpha)} \int_\tau^1 (1 - s)^{\alpha-p-1}ds + \frac{(\tau^{\alpha-1} - t^{\alpha-1})}{\Delta_1 \Gamma(\alpha - q)} \sum_{i=1}^N a_i \xi_i^{\alpha-q-1} \right]$$

$$= f^M a^M \left[\frac{|t^{\alpha-1} - \tau^{\alpha-1}|}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-p-1}ds + \frac{1}{\Gamma(\alpha)} \int_0^t \left|(\tau - s)^{\alpha-1} - (t - s)^{\alpha-1}\right|ds \right.$$

$$\left. + \frac{1}{\Gamma(\alpha)} \int_t^\tau (\tau - s)^{\alpha-1}ds + \frac{(\tau^{\alpha-1} - t^{\alpha-1})}{\Delta_1 \Gamma(\alpha - q)} \sum_{i=1}^N a_i \xi_i^{\alpha-q-1} \right]$$

$$= f^M a^M \left[\frac{(\tau^{\alpha-1} - t^{\alpha-1})}{\Gamma(\alpha)} \frac{1}{(\alpha - p)} + \frac{1}{\Gamma(\alpha)} \int_0^t \left[(\tau - s)^{\alpha-1} - (t - s)^{\alpha-1}\right]ds \right.$$

$$\left. + \frac{1}{\Gamma(\alpha)} \int_t^\tau (\tau - s)^{\alpha-1}ds + \frac{\tau^{\alpha-1} - t^{\alpha-1}}{\Delta_1 \Gamma(\alpha - q)} \sum_{i=1}^N a_i \xi_i^{\alpha-q-1} \right]$$

$$= f^M a^M \left[\frac{(\tau^{\alpha-1} - t^{\alpha-1})}{(\alpha - p)\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1}ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}ds \right.$$

$$\left. + \frac{(\tau^{\alpha-1} - t^{\alpha-1})}{\Delta_1 \Gamma(\alpha - q)} \sum_{i=1}^N a_i \xi_i^{\alpha-q-1} \right]$$

$$= f^M a^M \left[\frac{(\tau^{\alpha-1} - t^{\alpha-1})}{(\alpha - p)\Gamma(\alpha)} + \frac{\tau^\alpha}{\alpha\Gamma(\alpha)} - \frac{t^\alpha}{\alpha\Gamma(\alpha)} + \frac{(\tau^{\alpha-1} - t^{\alpha-1})}{\Delta_1 \Gamma(\alpha - q)} \sum_{i=1}^N a_i \xi_i^{\alpha-q-1} \right]$$

$$= f^M a^M (\tau^{\alpha-1} - t^{\alpha-1}) \left[\frac{1}{(\alpha - p)\Gamma(\alpha)} + \frac{1}{\Delta_1 \Gamma(\alpha - q)} \sum_{i=1}^N a_i \xi_i^{\alpha-q-1} \right]$$

$$+ f^M a^M \frac{(\tau^\alpha - t^\alpha)}{\Gamma(\quad)}$$

for $0 \le t < \tau \le 1$. Since the functions t^{α}, $t^{\alpha-1}$, are uniformly continuous on the interval $[0, 1]$, we have TU is an equicontinuous set. Further, $TU \subseteq U$ implies that T is uniformly bounded. Thus T is completely continuous.

Let $u, v \in [v_0, w_0]$ be such that $u \le v$. Then $v_0 \le u \le v \le w_0$. By (A1), we have

$$Tu(t) = \int_0^1 G(t, s)a(s)f(s, u(s))\, ds \le \int_0^1 G(t, s)a(s)f(s, v(s))\, ds = Tv(t),$$

thus, T is monotonically increasing in $[v_0, w_0]$.

We now prove that $v_0 = 0$ is a lower solution of T, that is, $v_0 \le Tv_0$. Indeed, for $v_0 \in P$, we have $Tv_0 \in P$ and so

$$Tv_0(t) \ge t^{\alpha-1} \int_0^1 J(s)a(s)f(s, v_0(s))\, ds$$

$$\ge t^{\alpha-1} \int_0^1 J(s)a(s)f(s, 0)\, ds$$

$$\ge 0 = v_0(t).$$

Finally, we show that $w_0 = R$ is an upper solution of T, that is, $Tw_0 \le w_0$. Clearly,

$$Tw_0(t) \le \int_0^1 J(s)a(s)f(s, w_0(s))\, ds \le R = w_0(t)$$

holds, so $w_0 = R$ is an upper solution of T.

Thus, if we construct sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ as

$$v_n = Tv_{n-1}, w_n = Tw_{n-1}, n = 1, 2, 3, \ldots$$

then

$$v_0 \le v_1 \le v_2 \le \ldots \le v_n \le w_n \le w_{n-1} \le \ldots \le w_1 \le w_0,$$

and $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ converges, respectively, to v and w, which are the smallest and greatest fixed points of T in $[v_0, w_0]$. By (3.5), we know that the zero function cannot be a solution of the problem (1.1)–(1.2). Thus, $\max_{0 \le t \le 1} v(t) \ge 0$ and $\max_{0 \le t \le 1} w(t) > 0$. Since $v \le w$, then Theorem 2.2 and Theorem 3.1 guarantee that v and w are the two positive nondecreasing solutions of the problem (1.1)–(1.2). This completes the proof of the theorem.

Remark 3.3 *If $f(t,0) = 0$ in (3.5), then the conclusion of Theorem 3.2 yields that (1.1)–(1.2) has at least one nondecreasing nonnegative solution and one nondecreasing positive solution.*

As an application of Theorem 3.2 and Remark 3.3, we consider the case where the nonlinear function f in (1.1) is a model of hematopoiesis (red blood production model), that is, we consider

$$D_{0+}^\alpha u(t) + \frac{u^l}{1 + u^m} = 0,\ t \in (0,1) \tag{3.6}$$

with the multipoint BCs (1.2). The following theorem provide a sufficient condition for the existence of three positive solutions to (3.6).

Theorem 3.4 *Let $m > l > 0$ and*

$$\frac{(m-l)}{m}\left(\frac{l}{m-l}\right)^{\frac{l-1}{m}}\int_0^1 a(s)J(s)\,ds \le 1, \tag{3.7}$$

then the BVP (3.6) together with the multipoint BCs (2.2) has at least one zero or non-negative solution and one positive solution.

Proof. Set $f(t,u) = \frac{u^l}{1+u^m}$. Clearly, $f'(u) = 0$ at $u = \left(\frac{l}{m-l}\right)^{1/m}$. Set $R = \left(\frac{l}{m-l}\right)^{1/m}$. Then $f'(u) \ge 0$ for $0 \le u \leq R$, $f'(u) \le 0$ for $0 \le u \geq R$, and $f'(u)00$ for $0 \le u = R$. Thus $f(t,u)$ is nondecreasing in $(0,R]$ and attains its maximum at $u = R$ and the maximum value is given by $\frac{(m-l)}{m}\left(\frac{l}{m-l}\right)^{\frac{l}{m}}$. Hence the condition (3.4) is satisfied if $\frac{(m-l)}{m}\left(\frac{l}{m-l}\right)^{\frac{l-1}{m}} \le \frac{1}{\int_0^1 a(s)J(s)\,ds}$, which is equivalent to (3.7). Since $f(t,0) = 0$, then the first solution $v_0(t)$ may be a zero solution or nonnegative solution, where as the second solution $w_0(t)$ is a positive solution. The theorem is proved.

The following example gives the existence of at least two positive solutions.

Example 3.5 *Consider the problem*

$$D_{0+}^{5/2}x(t) + \frac{1}{8}\left[\frac{t(1-t)^{1/2}}{\Gamma(5/2)} + \frac{2\sqrt{2}}{3}\mathfrak{g}(t)\right]^{-1}(e^{x(t)} + e^{y^2(t)}) = 0,\ t \in (0,1), \tag{3.8}$$

with the multipoint BCs

$$x(0) = x'(0) = 0,\quad x'(1) = \frac{1}{\sqrt{2}}x'(1/2). \tag{3.9}$$

Here $m = 3, \alpha = 5/2, p = 1, q = 1, M = 1, \xi_1 = 1/2\ a_1 = 1/\sqrt{2}$,

$$a(t) = \left[\frac{t(1-t)^{1/2}}{\Gamma(5/2)} + \frac{2\sqrt{2}}{3}\mathfrak{g}(t)\right]^{-1},\ 0 < t < 1,$$

$$\mathfrak{g}(t) = \frac{1}{\Gamma(5/2)} \begin{cases} \frac{1}{\sqrt{2}}(1-t)^{1/2} - (1/2-t)^{1/2}, & t \leq 1/2 \\ \frac{1}{\sqrt{2}}(1-t)^{1/2}, & 1/2 \leq t \leq 1, \end{cases}$$

$$f(t, x, y) = \frac{1}{8}e^{x(t)} \ t \in [0, 1].$$

Clearly, $f(t, 0) \neq 0$ and $\int_0^1 J(t)a(t)\,dt = 1$. Hence by Theorem 3.2 and Remark 3.3, (3.8)–(3.9) has at least two positive nondecreasing solutions.

References

[1] H. Amann, Fixed point equations and nonlinear eigen value problem in ordered Banach spaces, *SIAM Rev.* **18** (1976), 620 - 709.

[2] H. Amann, On the number of solutions of nonlinear equations in ordered Banck spaces, *J. Funct. Anal.* **11** (1972), 348 - 384.

[3] A. Granas and J. Dugundji, Fixed Point Theory, *Springer-Verlag, New York* (2003).

[4] J. Henderson and R. Luca, Existence of postive solutions for a singular fractional boundary value problem, *Nonl. Anal: Model. Control* **22**(1) (2017), 99 - 114.

[5] J. Henderson and R. Luca, Systems of Riemann- Liouville fractional equations with multipoint boundary conditions, *Appl. Math. Comput.* **309** (2017), 303 - 323.

[6] J. Henderson and R. Luca , Positive solutions for a system of coupled fractional boundary value problems, *Lithuanian Mathematical Journal* **58**(1) (2018), 15 - 32. DOI 10.1007/s10986-018-9385-4.

[7] R. Luca, Positive solutions for a system of Riemann-Liouville fractional differential equations with multi-point fractional boundary conditions, *Bound. Value Probl.* **102** (2017) (2017).

[8] Seshadev Padhi and Smita Pati, Positive solutions of a fractional differential equation with nonlinear Rieman-Stieltjes type boundary conditions, *PanAmerican Math. J.* **27**(4) (2017), 100 - 107.

[9] Seshadev Padhi and Sri Rama Vara Prasad Bhuvanagiri, Monotone Iterative Method for Solutions of a Cantilever Beam Equation with One Free End, *Advances in Nonlinear Variational Inequalities* **23**(2) (2020), 15 - 22.

[10] Seshadev Padhi, John R.Graef and Smita Pati Multiple positive solutions for a boundary value problem with non linear nonlocal Riemann-Stieltjes integral boundary conditions, *Frac. Calc. Appl. Anal.* **21**(3) (2018), 716 - 745.

[11] Seshadev Padhi, B. S. R. V. Prasad and Divya Mahendru, System of Riemann-Liouville fractional differential equations with nonlocal boundary conditions - Existence, uniqueness and multiplicity of solutions, *Mathematical Methods in the Applied Sciences* https://doi.org/10.1002/mma.5812 (2019).

[12] Seshadev Padhi, B. S. R. V. Prasad and Divya Mahendru, System of Riemann-Liouville fractional differential equations with nonlocal boundary conditions - Method of fixed point index, *Mathematical Methods in the Applied Sciences* https://doi.org/10.1002/mma.5931 (2019).

[13] J. Wang, H. Xiang and Z. Liu, Positive solution to nonzero boundary value problem for a coupled system of nonlinear fractional differential equations, *Int. J. Diff. Eqn.* Article ID 186926, 12pp. (2010).

[14] S. Xie and Y. Xie , Positive solutions of higher order nonlinear fractional differential systems with nonlocal boundary conditions, *Journal of Apllied Analyis and Computation* **6**(4) (2016), 1211 - 1227.