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Monotone Iterative Method for Solutions of a Cantilever Beam Equation with One Free End

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Abstract

In this paper, we apply the monotone iterative method for the existence of two positive solutions of fourth-order two-point boundary value problem

$$\begin{aligned} u''''(t) &= f(t, u(t)), \quad 0 < t < 1, \\ u(0) =& u'(0) = u''(1) = u'''(1) = 0, \end{aligned}$$

which models a cantilever beam equation, where one end is kept free. Here $f \in \mathcal{C}([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$. The sufficient condition is interesting, new and easy to verify. Our condition do not require any super-linearity or sub-linearity conditions on the function f at 0 or ∞. Our result uses the monotonically increasing property of f in a certain interval to prove the existence of a positive solution(s). This approach completely differs from the existing results in the literature. An example is presented at the end to illustrate the usefulness of our result. Although our theorem predicts the existence of two positive solutions, our example and the iterative scheme constructed in the example shows that the two iterative sequences of solutions converge to a single solution.

AMS Subject Classification: 34B10, 34B18, 65L10

Key Words and Phrases: Cantilever beam, Boundary value problems, Monotone iterative method, Positive solutions.

1 Introduction

In this work, we are interested in demonstrating the use of monotone iteration method for studying the existence of two positive solutions of the nonlinear fourth order two point boundary value problem (BVP)

$$\begin{aligned} x''''(t) &= f(t, x(t)), \quad t \in [0,1], \\ x(0) = &x'(0) = x''(1) = x'''(1) = 0, \end{aligned} \tag{1.1}$$

which is a cantilever beam equation describing the models of deflection of an elastic beam fixed at left end and freed at the other end. Here, $f \in \mathcal{C}([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$.

Boundary value problems of type (1.1) with various nonlinearities on f, have been studies by many authors [4, 7, 8, 10, 11, 12, 13, 14, 15, 16]. The methods used in [4, 7, 8, 10, 11, 12, 13, 14, 15, 16] are contracting mapping principle, iterative method, fixed point index theory in cones, Krasnosel'skii fixed point theorem, lower and upper solution method and degree theory.

Equation (1.1) is a particular case of a (n, p) boundary value problem, studied by Agarwal and Regan [1, 2]. If we choose $n = 4$ and $p = 2$, then the problem considered in [1, 2], reduces to the problem (1.1). The results in [1, 2] are completely based on Krasnosel'skii fixed point theorem in a cone.

Our results in this paper are completely different from the approach by the authors in [4, 7, 8, 10, 11, 12, 13, 14, 15, 16]. We shall apply the monotone iterative method to obtain sufficient conditions on the existence of one positive solution of (1.1), and an iterative scheme for approximating the solutions. The monotone iteration method we shall use in this paper, is imported from [5, 6, 9, 17]. Our work is motivated by a recent work of Zhang [18], who used iteration method for existence of monotone positive solutions of an elastic beam equation with a different boundary conditions.

The following theorem states the main result of this paper.

Theorem 1.1. *If there exists constants r and R with $0 < r < R$ such that*

$$\textit{\textbf{(A1)}:}\ 6r \le f(t,u) \le f(t,v) \le 6R \text{ for } \frac{1}{36}r \le u \le v \le R \text{ and } \frac{1}{2} \le t \le 1, \tag{1.2}$$

then the problem (1.1) has two positive solutions.

Remark 1.2. *Although Theorem 1.1 claims, under the assumption $(A1)$, that the problem (1.1) has two positive solutions, they may be equal also, as we shall see in our Example 3.1.*

Remark 1.3. *One can observe from the assumption $(A1)$ that, we do not require any super-linearity or sub-linearity on f either at 0 or ∞. The only assumption we require on f is that f must be monotonically nondecreasing in the subinterval $[1/2, 1]$. The function f may decrease or non-decrease or indentically zero on the other half of the interval $[1/2, 1]$.*

This shows that our assumption (A1) is not comparable with the results in [1, 2, 3, 4, 7, 8, 10, 11, 12, 13, 14, 15, 16]. Our monotone iterative method is different from the ones used in a recent paper due to Dang and Qay [8], and the conditions on f are also different.

2 Preliminaries

In this section, we provide some basic concepts monotone iteration method. We shall use the following well-known monotone iterative method (see [5, 6, 9] or Theorem 7.A in [17]).

Theorem 2.1. *Let X be a real Banach space and K be a cone in X. Assume that there exist constants v_0 and w_0 with $v_0 \leq w_0$ and $[v_0, w_0] \subset X$ such that*

(i) $T : [v_0, w_0] \to X$ *is completely continuous;*

(ii) T *is a monotonic increasing operator on* $[v_0, w_0]$*;*

(iii) v_0 *is a lower solution of* T*, that is,* $v_0 \leq Tv_0$*;*

(iv) w_0 *is an upper solution of* T*, that is,* $Tw_0 \leq w_0$*.*

Then T has a fixed point and the iterative sequence $v_{n+1} = Tv_n$ and $w_{n+1} = Tw_n$, $n = 1, 2, 3, \ldots$ with

$$v_0 \leq v_1 \leq v_2 \leq \ldots \leq v_n \leq \ldots \leq w_n \leq w_{n-1} \leq \ldots \leq w_1 \leq w_0$$

converges to v and w respectively, which are the greatest and lowest fixed points of T in $[v_0, w_0]$.

In this paper, we set $X = \mathcal{C}[0, 1]$ to be the Banach space with standard norm

$$\|x\| = \max_{0 \leq t \leq 1} |x(t)|.$$

Define a cone K on X by

$$K = \{x \in \mathcal{C}[0, 1] : x(t) \geq 0, t \in [0, 1]\},$$

and an operator $T : K \to K$ by

$$Tx(t) = \int_0^1 G(t, s) f(s, x(s))\, ds, \tag{2.1}$$

where $G(t, s)$ is the Green's kernel, given by

$$G(t, s) = \frac{1}{6} \begin{cases} s^2(3t - s), & 0 \leq s \leq t \leq 1, \\ t^2(3s - t), & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Let $g(s) = \frac{s^2}{2}$ and $c(t) = \frac{2}{3}t^2$. From a straight forward calculation, also proved in [12], one can show that $G(t,s)$ satisfies the inequality

$$c(t)g(s) \leq G(t,s) \leq g(s) \text{ for } 0 \leq t, s \leq 1. \tag{2.3}$$

Since it is useful to work on a smaller cone than K, we consider a cone K_1 of the type

$$K_1 = \{x \in K : x(t) \geq 0 \text{ and } \min_{t\in[0,1]} x(t) \geq c_{a,b}\|x\|\},$$

where $c_{a,b} = \min_{t\in[a,b]} c(t)$ and $[a,b]$ is an arbitrary subinterval of $[0,1]$ with $c_{a,b} > 0$. For $[a,b] \subset [0,1]$, if $c_{a,b} > 0$, the condition (2.3) ensures that T maps K into K_1. Since (2.3) is valid for any $t \in [0,1]$, we can work on the subinterval $[1/2,1] \subset [0,1]$, for which the inequality

$$\frac{1}{6} \cdot \frac{s^2}{2} \leq G(t,s) \leq \frac{s^2}{2} \tag{2.4}$$

replaces (2.3), where $\min_{t\in[1/2,1]} = \min_{t\in[1/2,1]} \frac{2}{3}t^2 = \frac{1}{6} := \mu$. In this case, the operation T, defined in (2.1), maps the cone K into the subcone P, where

$$P = \{x \in \mathcal{C}[0,1] : \min_{t\in[1/2,1]} x(t) \geq \frac{1}{6}\|x\|\}. \tag{2.5}$$

Clearly, $x(t)$ is a positive solution of the problem (1.1) if and only if $x(t)$ is a fixed point of the operator T on the subcone P.

3 Proof of Theorem 1.1

In order to prove our theorem, we shall consider the cone P, defined in (2.5). Let $x \in P$. Then

$$\|Tx\| \leq \frac{1}{2} \int_0^1 s^2 f(s, x(s))\, ds$$

and

$$\min_{t\in[1/2,1]} Tx(t) \geq \left(\min_{t\in[1/2,1]} \frac{2}{3}t^2\right) \int_0^1 \frac{1}{2}s^2 f(s,x(s))\, ds = \frac{1}{6} \cdot \frac{1}{2} \int_0^1 s^2 f(s,x(s))\, ds \geq \frac{1}{6}\|Tx\|$$

implies that $T : P \to P$. Also, T is well-defined. Set $v_0 = \frac{1}{36}r$ and $w_0 = R$; then $v_0 \leq w_0$. By the continuity of $f(t,x(t))$ and $G(t,s)$ for $t,s \in [0,1]$, it is easy to show that $T : [v_0, w_0] \to P$ is completely continuous.

Let $u, v \in [v_0, w_0]$ be such that $u \leq v$. Then $v_0 \leq u \leq v \leq w_0$. By (A1), we have

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s))\, ds \leq \int_0^1 G(t,s) f(s,v(s))\, ds = Tv(t),$$

thus, T is monotonically increasing in $[v_0, w_0]$.

We now prove that $v_0 = \frac{1}{36}r$ is a lower solution of T, that is, $v_0 \le Tv_0$. Indeed, for $v_0 \in P$, we have $Tv_0 \in P$ and so

$$\begin{aligned} Tv_0(t) &\ge \frac{1}{6}\|Tv_0(t)\| \ge \frac{1}{6} \cdot \min_{t\in[1/2,1]} Tv_0(t) \\ &\ge \frac{1}{36} \cdot \int \frac{s^2}{2} f(s, v_0(s))\, ds \\ &\ge \frac{1}{36} \cdot 6r \cdot \int_0^1 \frac{s^2}{2}\, ds = \frac{1}{36}r = v_0(t). \end{aligned}$$

Finally, we show that $w_0 = R$ is an upper solution of T, that is, $Tw_0 \le w_0$. Clearly,

$$Tw_0(t) \le \int_0^1 \frac{s^2}{2} f(s, w_0(s))\, ds \le R = w_0(t)$$

holds, so $w_0 = R$ is an upper solution of T.

Thus, if we construct sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ as

$$v_n = Tv_{n-1}, w_n = Tw_{n-1}, n = 1, 2, 3, \ldots$$

then

$$v_0 \le v_1 \le v_2 \le \ldots \le v_n \le w_n \le w_{n-1} \le \ldots \le w_1 \le w_0,$$

and $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ converges, respectively, to v and w, which are the greatest and smallest fixed points of T in $[v_0, w_0]$. Since $v \le w$, then Theorem 2.1 guarantees that v and w are the two positive solutions of the problem (1.1). This completes the proof of the theorem.

In the following, we provide a simple example to illustrate our Theorem 1.1.

Example 3.1. *Consider the boundary value problem*

$$x''''(t) = f(t, x(t)), 0 < t < 1,\ x(0) = x'(0) = x''(1) = x'''(1) = 0 \tag{3.1}$$

where

$$f(t, x) = \frac{1}{2}\left(35 + exp\left(-\frac{1}{x+2}\right)\right) \tag{3.2}$$

For $u \le v$, we have $\exp\left(-\frac{1}{u+2}\right) \le \exp\left(-\frac{1}{v+2}\right)$, which implies that $f(t, u) \le f(t, v)$ for $u \le v$. Set $r = \frac{8}{3}$ and $R = \frac{10}{3}$; then for $\frac{1}{2} \le t \le 1$ with $\mu = \frac{1}{6}$, we have

$$f(t, u) \ge \frac{35}{2} = 17.5 > 16 = 6r \text{ for } u \ge \frac{2}{27} = \frac{1}{36}r,$$

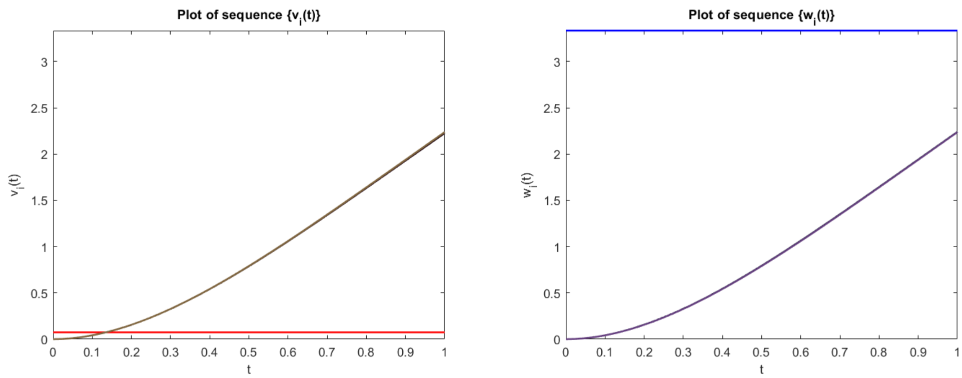

Figure 1: Plots of the sequences $\{v_i(t)\}$ and $\{w_i(t)\}$.

Table 1: Table demonstrating the converging nature of sequences $\{v_i(t)\}$ and $\{w_i(t)\}$

$v_i(t)\backslash t$	0	0.2	0.4	0.6	0.8	1
$v_0(t)$	0.074074074	0.074074074	0.074074074	0.074074074	0.074074074	0.074074074
$v_1(t)$	0	0.15552957	0.541385374	1.057838528	1.633654112	2.226091178
$v_2(t)$	0	0.155579145	0.54225545	1.060915579	1.640165231	2.236830437
$v_3(t)$	0	0.155579175	0.542256831	1.060922621	1.640182382	2.236860034
$v_4(t)$	0	0.155579175	0.542256833	1.060922637	1.640182427	2.236860115
$v_5(t)$	0	0.155579175	0.542256833	1.060922637	1.640182427	2.236860115

$w_i(t)\backslash t$	0	0.2	0.4	0.6	0.8	1
$w_0(t)$	3.333333333	3.333333333	3.333333333	3.333333333	3.333333333	3.333333333
$w_1(t)$	0	0.156453427	0.544601243	1.064122165	1.643358136	2.23931432
$w_2(t)$	0	0.155579691	0.542260551	1.060929946	1.640190773	2.23686686
$w_3(t)$	0	0.155579175	0.542256839	1.060922654	1.640182449	2.236860134
$w_4(t)$	0	0.155579175	0.542256833	1.060922637	1.640182427	2.236860115
$w_5(t)$	0	0.155579175	0.542256833	1.060922637	1.640182427	2.236860115

and

$$f(t,v) \leq \frac{1}{2}\left(35+\exp\left(-\frac{1}{v+2}\right)\right) \leq 18 < 20 = 6R \text{ for } v \leq R = \frac{10}{3},$$

which implies that the condition (A1) of Theorem 1.1 is satisfied. Thus by Theorem 1.1, the problem (3.1), with f given in (3.2) has at least two positive solutions.

Starting with $v_0(t) = \frac{r}{36} = 0.074074074$ and $w_0(t) = R = 3.333333333$ and constructing the sequences $v_i(t) = Tv_{i-1}(t)$ and $w_i(t) = Tw_{i-1}(t)$ for $i = 1, 2, 3, 4, 5,$ we found that the sequence $\{v_i\}$ and $\{w_i\}$ converges (cf. Table 1) to the solution (cf. Fig 1).

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