Nonlinear singular boundary value problems

Many boundary value problems arising in physical science and applied mathematics involve ordinary differential equations subject to two-point boundary conditions. In general, it is very difficult to obtain analytical solutions of nonlinear boundary value problems. More difficulties arise when we deal with nonlinear singular boundary value problems. The solutions of these problems are not trivial because of the singularity. The one dimensional nonlinear singular differential equation is given by

$$\frac{1}{q(x)}\big(p(x)y'(x)\big)' = f(x, y(x)), \qquad ' \equiv \frac{d}{dx} \tag{1}$$

with appropriate boundary conditions. The singular problem (2) with $p = q = x^\alpha$ where $\alpha = 0, 1, 2$ arises in the study of various *tumor growth problems* with $f(x, y) = \frac{\theta y}{y+\kappa}$, where θ, κ are positive constants and $\alpha = 2$ in the study of steady-state oxygen diffusion in a spherical cell with Michaelis-Menten uptake kinetics [Lin, 1976]. A similar problem arises in modelling of heat conduction in human head [Gray, 1980] with $\alpha = 2$ and $f(x, y) = -\delta e^{-\theta y}$ where θ, δ are positive constants.

The time dependent Emden-Fowler equations is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\alpha}{x}\frac{\partial u}{\partial x} + af(x,t)F(u) + h(x,t) = \frac{\partial u}{\partial t},\ 0 < x < l \tag{2}$$

with appropriate boundary conditions. Here u is the temperature, $f(x,t)F(u) + h(x,t)$ is the nonlinear heat source and t is the dimensionless time variable. For the steady-state case with $h(x,t) = 0$, Eq. (2) is the Emden-Fowler equation given by

$$u_{xx} + \frac{\alpha}{x}u_x + af(x)F(u) = 0.$$

For $f(x) = 1$ and $F(u) = u^n$, this equation is known as the standard Lane-Emden equation of the first kind, whereas the second kind is obtained when $F(u) = e^u$. It is well-known that the Lane–Emden equation is used in modelling a thermal explosion in either an infinite cylinder ($\alpha = 1$) or a sphere ($\alpha = 2$), where α is the shape factor of the equation.
Analytical results are available only for a limited number of simple problems and therefore numerical or semi-numerical methods are frequently needed to solve considered models. There are several numerical schemes in the literature for solving such models. Unfortunately, most of these existing techniques involve a huge amount of computational work which require root-finding techniques, linearization, discretization and a need of solving a large system of nonlinear equations to get a numerical solutions. This led to a clear motivation for the development of some new numerical or semi-numerical schemes.

Publication under this topic:

1. ***Randhir Singh***, S. Singh A.M. Wazwaz, A modified homotopy perturbation method for singular time dependent Emden-Fowler equations with boundary conditions, ***Journal of Mathematical Chemistry*** 54, 918–931 (2016).

2. ***Randhir Singh***, A.M. Wazwaz, Numerical solution of the time dependent Emden-Fowler equations with boundary conditions using modified decomposition method, ***Applied Mathematics and Information Sciences*** 10, 403–408 (2016).

3. **R. Singh**, A.M. Wazwaz, J. Kumar, An efficient semi-numerical technique for solving nonlinear singular boundary value problems arising in various physical models, ***International Journal of Computer Mathematics*** 93, 1330-1346 (2016).

4. Nilima Das, ***Randhir Singh***, A.M. Wazwaz, J. Kumar: An algorithm based on the variational iteration technique for the Bratu-type and the LaneEmden problems, ***Journal of Mathematical Chemistry*** 54, 527-551 (2016).

5. ***Randhir Singh***, J. Kumar, An efficient numerical technique for the solution of nonlinear singular boundary value problems, ***Computer Physics Communications***185, 1282–1289 (2014).

6. ***Randhir Singh***, J. Kumar, G. Nelakanti, Approximate series solution of singular boundary value problems with derivative dependence using Green's function technique, ***Computational and Applied Mathematics*** 33, 451–467 (2014).

7. ***Randhir Singh***, J. Kumar: The Adomian decomposition method with Green's function for solving nonlinear singular boundary value problems, ***Journal of Applied Mathematics and Computing*** 44, 397–416 (2014).

8. ***Randhir Singh***, J. Kumar, G. Nelakanti: Numerical solution of singular boundary value problems using Green's function and improved decomposition method, ***Journal of Applied Mathematics and Computing*** 43, 409–425(2013).

9. ***Randhir Singh***, J. Kumar: Computation of eigenvalues of singular Sturm-Liouville problems using modified Adomian decomposition method, ***International Journal of Nonlinear Science*** 15, 247–258 (2013).