DEPARTMENT OF MATHEMATICS

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MA107 Mathematics-II Session: SP/ 2019

Tutorial-2

Module -III

Q10.Form a partial differential equation by the method of elimination of arbitrary constants from the following:

(a) $z=a(x+y)+b(x-y)+abt+c$ where z, x, y and t are variables and others are constants.
(b) $z=ax^3+by^3$
(c) $\log(az-1)=x+ay+b$.
(d) $\frac{x}{a}+\frac{y}{b}-\frac{z}{ab}=0$

Q11.Form a partial differential equation by the method of elimination of arbitrary functions from the following:

(a) $xyz=f(x+y+z)$
(b) $z=f(x^2-y^2)$
(c) $F(x+y+z,x^2+y^2+z^2)=0$
(d) $F(ax+by+cz,x^2+y^2+z^2)=0$

Q12.Solve the partial differential equation:

(a) $(x^2+y^2+z^2)p-2xyq=-2xz$
(b) $pz-qz=z^2+(x+y)^2$
(c) $(y+z)p+(z+x)q=x+y$
(d) $(mz-ny)p+(nx-lz)q=ly-mx$
(e) $p+3q=5z+\tan(y-3x)$

Q13.Using the method of separation of variables, solve the following equation $\frac{\partial^2 u}{\partial x \partial t}=e^{-t}cosx$ given that

$u=0$ when $t=0$ and $\frac{\partial u}{\partial t}=0$ when $x=0$.

Q14.Using the method of separation of variables, solve the following equation $4\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=3u$ and $u=e^{-5y}$

when $x=0$.

Q15.Solve the heat and wave equations by the method of separation of variables.

Q16.Find the temperature in a bar of length 2 whose ends kept at zero and lateral surface insulated if the initial temperature is $\sin\frac{\pi x}{2}+3\,sin\frac{5\pi x}{2}$.

Q17.A tightly stretched string with fixed end points $x=0$ and $x=\pi$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03\ sinx - 0.04\ sin3x$ then find the displacement *y(x,t)* at any point of string at any time *t*.

Q18.The temperature distribution in a bar of length π, which is perfectly insulated at the ends *x=0* and *x=π* is governed by the partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ .Assuming the initial temperature as *u(x,0)=f(x)=cos2x*, find the temperature distribution at any instant of time.

Q19.Solve the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, representing the vibration of a string of length l, fixed at both ends, given that *y(0,t)=0, y(1,t)=0; y(x,0)=f(x)* and $\frac{\partial}{\partial t} y(x,0) = 0{,}0 < x < l$.

Module IV

Q.1

(a) Show that $\lim_{z\to 0} \frac{\bar{z}^2}{z} = 0$

(b) Prove that $\lim_{z\to z_0} \bar{z} = \overline{z_0}$

(c) Show that the limit of the function $f(z) = (\frac{z}{\bar{z}})^2$ as z tends to 0 does not exist.

Q.2

(a) Prove that $f(z)=z^2$ is continuous at $z=z_0$.

(b) Is the function $f(z)=\frac{3Z^4-2Z^3+8Z^2-2Z+5}{Z-i}$ continuous at $z=i$.

Q.3

(a) Show that $f(z)=\bar{z}$ is non-analytic anywhere.

(b) Show that continuity does not imply differentiability by considering the function $f(z) = |z|^2$.

(c) Using the definition, find the derivative of $w=f(z)=z^3-2z$ at the point where
(I) $z=z_0$ (II) $z=-1$.

Q4.Show that each of these functions is nowhere analytic:

(a) $f(z)= xy + iy$ *(b)* $f(z)=2xy+i(x^2-y^2)$ *(c)* $f(z)= e^y e^{ix}$

Q5.Verify whether the function $f(z) = \begin{cases} \frac{x^3 y(y-ix)}{x^6+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is non - analytic at $z=0$.

Q6.Let *u(x,y)* and *v(x,y)* denote the real and imaginary components of the function *f* (z) defined by

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} \text{ when } z \neq 0\,, 0 \text{ when } z = 0 \end{cases}$$

Verify that the Cauchy-Riemann equations are satisfied at the origin *z=(0,0).*

Q7.Show that *u(x,y)* is harmonic in some domain and find a harmonic conjugate *v(x,y)* when

(I) $u(x,y)=2x(1-y)$ (II) $u(x,y)=2x-x^3+3xy^2$

Q8.Derive Cauchy Riemann partial differential equations for the necessary conditions of analyticity of a function of complex variable. When these conditions become sufficient?

Q9.Derive Cauchy Reimann equations in polar form and prove that

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \text{ and } \frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}=0$$

Q10.If a function *f(z)=u(x,y)+iv(x,y)* is analytic in a domain *D*, then its component functions *u* and *v* are harmonic in *D*.

Q11.State and prove Cauchy's Integral Theorem.

Q12.State and prove Cauchy's Integral Formula.

Q13.For the functions *f* and contour *C*, use parametric representations for C to evaluate integral *f(z)* , where *f(z)=(z+2)/z* and *C* is

(a) The semicircle $z=e^{i\theta}$ $(0\leq\theta\leq\pi)$ (b) the circle $z=2e^{i\theta}(0\leq\theta\leq2\pi)$.

Q14.Apply Cauchy's theorem to show that $\int f(z)dz$=0 when the contour *C* is the unit circle $|z|=1$, in either direction, and when

(a) $f(z)=\frac{z^2}{z-3}$ (b) $f(z)=ze^{-z}$ (c) $f(z)=\frac{1}{z^2+z+2}$

Q15.Let *C* denote the positively oriented boundary of the square whose sides lie along the lines *x=±2* and *y=±2*.Evaluate each of these integrals:

(a) $\oint \frac{e^{-z}}{z-(\frac{\pi i}{2})}$ (b) $\oint \frac{zdz}{2z+1}$ (c) $\oint \frac{cosz\ dz}{z(z^2+8)}$

Q16.Find the value of the integral of *g(z)* around the circle *|z-i| =2* in the positive sense when

(a) g(z)=$\frac{1}{z^2+1}$ (b) g(z)=$\frac{1}{(z^2+4)^2}$

Q17.Show that when *0<|z-1|<2,* $\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}$.

Q18.Write down the Laurent series in powers of *z* that represent the function

f(z)=$\frac{1}{z(1+z^2)}$ in the domain *0<|z|<1*.

Q19. Find the residue at *z=0* of the function

(a) $\frac{1}{z+z^2}$ (b) $\frac{z-sinz}{z}$ (c) $\frac{cotz}{z^4}$

Q20.Use Cauchy's residue theorem to evaluate the integrals of each of these functions around the circle *|z|=3* in the positive sense:

(a) $\frac{\exp(-z)}{z^2}$ (b) $\frac{z+1}{z(z-2)}$ (c) $z^2exp(1/z)$

Q21.Find the value of the integral $\int_C \frac{3z^3+2\ dz}{(z-1)(z^2+9}$ taken counterclockwise around the circle

(a) *|z-2|=2* (b) *|z|=4*.

Q22.Evaluate the improper integral using residues :

(a) $\int_0^{\infty} \frac{xsin2xdx}{x^2+3}$ (b) $\int_0^{\infty} \frac{cosax}{x^2+1} dx$ (a>0)

Q23.Use residues to evaluate the improper integrals:

(a) $\int_{-\infty}^{\infty} \frac{xsinax}{x^2+4} dx$ (a>0) (b) $\int_{-\infty}^{\infty} \frac{xsinxdx}{(x^2+1)(x^2+4)}$

Q24. Use residue to evaluate the definite integrals

(a) $\int_{0}^{2\pi} \frac{d\theta}{1+acos\theta}$ (-1<a<1) (b) $\int_{0}^{\pi} \frac{d\theta}{(a+cos\theta)^2}$ (a>1) (c) $\int_{0}^{2\pi} \frac{cos^2 3\theta d\theta}{5-4cos2\theta}$

1. Let X be a random variable with probability density function

$$f(x) = k(2x - 1) \text{ for } x = 1, 2, 3 \ldots, 12$$

for some constant k. What is the value of k. Also, find the cumulative distribution of X.

2. What is the probability density function of the random variable X whose cumulative distribution function is

$$F(x) = \begin{cases} 0 & \text{if } x < 2 \\ 0.5 & \text{if } 2 \le x < 3 \\ 0.7 & \text{if } 3 \le x < \pi \\ 1.0 & \text{if } x \ge \pi \end{cases}$$

3. Find the probability density function of the random variable X whose cumulative distribution function is

$$F(x) = \begin{cases} 0.00 & \text{if } x < -1 \\ 0.25 & \text{if } -1 \le x < 1 \\ 0.50 & \text{if } 1 \le x < 3 \\ 0.75 & \text{if } 3 \le x < 5 \\ 1.0 & \text{if } x \ge 5. \end{cases}$$

Also, find

a) $P(X \le 3)$ b) $P(X = 3)$ c) $P(X < 3)$

4. Let X be a random variable with probability density function

$$f(x) = \frac{2c}{3^x} \text{ for } x = 1, 2, 3 \ldots$$

for some constant c. What is the probability that X is even

5. The random variable X has density function

$$f(x) = \begin{cases} (k+1)x^2 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

find the value of constant k.

6. A random variable X has a cumulative distribution function

$$F(x) = \begin{cases} \frac{x}{2} & \text{if } 0 < x \le 1 \\ x - \frac{1}{2} & \text{if } 1 < x \le \frac{3}{2} \end{cases}$$

a) $P(X \le 0.5)$ b) $P(X \ge 0.5)$ c) $P(X \le 1.25)$ d) $P(X = 1.25)$

7. Let the distribution of X for $x > 0$ be

$$F(x) = 1 - \sum_{k=0}^{3} \frac{x^k e^{-x}}{k!}$$

What is the density function of X for $x > 0$

8. Let X is a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } 7 < x < 9 \\ 0 & \text{otherwise} \end{cases}$$

Find the cummulative distribution function of X.

9. Let X be a random variable with cumulative distribution function

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

what is the $P(0 \leq e^X \leq 4)$.

10. Let X be a random variable with probability density function

$$f(x) = \begin{cases} \frac{x}{8} & \text{for } x = 1, 2, 5 \\ 0 & \text{otherwise.} \end{cases}$$

Find

a) $E(X)$ b) $Var(X)$ c) $E(2X + 3)$ d) $Var(2X + 3)$

11. Let X is a random variable with density function

$$f(x) = \begin{cases} \theta x + \frac{3}{2}\theta^{\frac{3}{2}} x^2 & \text{for } 0 < x < \frac{1}{\sqrt{\theta}} \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$. What is the expected value of X.

12. Let X is a random variable with density function

$$f(x) = \begin{cases} 1.4e^{-2x} + 0.9e^{-3x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

then what is the expected value of X.

13. If the moment generating function for the random variable X is $M(t) = \frac{1}{1+t}$, what is the third moment of X about the point $x = 2$.

14. If the moment generating function for the random variable X is $M(t) = k(2 + 3e^t)^4$, what is the value of k.

15. If the moment generating function for the random variable X is

$$M(t) = 1 + 2t + 4t^2 + 8t^3 + 16t^4 + \ldots,$$

what is the third moment of X about the its mean.

16. Let X be a random variable with density function

$$f(x) = \begin{cases} ae^{-ax} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $a > 0$. if $M(t)$ denotes the moment generating function of X, what is $M(-3a)$.

17. Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{2x}{k^2} & 0 \le x \le k \\ 0 & \text{otherwise} \end{cases}$$

For what value of k is the variance of X equal to 8.

18. What is the probability of getting exactly 3 heads in 5 flips of a fair coin.

19. On six successive flips of a fair coin, what is the probability of observing 3 heads and 3 tails.

20. What is the probability that in 3 rolls of a pair of six-sided dice, exactly one total of 7 is rolled.

21. In a family of 4 children, what is the probability that there will be exactly two boys.

22. A Quiz is conducted which is consisting of 20 MCQs (multiple choices questions) with every MCQ having its four options out of which only one is correct. A person answered randomely all the questions. What is the probability (a) exactly 5 questions are correct (b) exactly 10 questions are correct.

23. A random variable X has a Poisson distribution with a mean of 3. What is the probability that X is bounded by 1 and 3, that is $P(1 \le X \le 3)$.

24. The number of traffic accidents per week in a small city has a Poisson distribution with mean equal to 3. What is the probability of exactly 2 accidents occur in 2 weeks.

25. A call center receives an average of 4.5 calls every 5 minutes. Each agent can handle one of these calls over the 5 minute period. If a call is received, but no agent is available to take it, then that caller will be placed on hold. Assuming that the calls follow a Poisson distribution, what is the minimum number of agents needed on duty so that calls are placed on hold at most 10% of the time.

26. If X is any random variable with mean μ and variance $\sigma^2 > 0$, then what are the mean and variance of the random variable $Y = \frac{X - \mu}{\sigma}$.

27. If $X \sim N(\mu, \sigma^2)$, then the random variable $Y = \frac{X - \mu}{\sigma} \sim N(0, 1)$.

28. If X be a normal distribution with mean 2 and variance 4, then find the $P(X > 2)$.

Answer

1. $k = \frac{1}{144}$, $F(1) = \frac{1}{144}$, $F(2) = \frac{4}{144}$, $F(3) = \frac{9}{144}$, $\ldots$, $F(12) = 1$.
2. $f(2) = 0.5$, $f(3) = 0.2$, $f(\pi) = 0.3$
3. $f(-1) = f(1) = f(3) = f(5) = 0.25$, $P(X \leq 3) = 0.75$, $P(X = 3) = 0.25$, $P(X < 3) = 0.5$
4. $c = 1$, $P(\text{X is even}) = \frac{1}{4}$.
5. $k = 2$.
6. $P(X \leq 0.5) = 0.25$, $P(X \geq 0.5) = P(X \leq 1.25) = 0.75$, $P(X = 1.25) = 0$
7. $\frac{1}{6}x^3e^{-x}$
8. $F(x) = \begin{cases} 0 & x \leq 7 \\ \frac{x}{2} & 7 < x < 9 \\ 1 & x \geq 9 \end{cases}$
9. $\frac{3}{4}$
10. $(a)3.75$, $(b)2.6875$, $(c)10.5$, $(d)10.75$
11. $\frac{17}{24\sqrt{\theta}}$
12. $\frac{9}{20}$
13. -38
14. $\frac{1}{625}$
15. 38
16. $\frac{1}{4}$
17. 6 because $k \geq 0$
18. $\frac{5}{16}$
19. $\frac{5}{16}$
20. $\frac{25}{72}$
21. $\frac{3}{8}$
22. (a) 0.2023, (b) 0.0099
23. $12e^{-3}$
24. $18e^{-6}$
25. 7
26. $E(Y) = 0$, $Var(Y) = 1$
28. 0.5