BIRLA INSTITUTE OF TECHNOLOGY

MESRA – 835215, RANCHI, INDIA

UG

Name: .... Roll No.: ....

Branch: .... Signature of Invigilator: ....

Semester: IVth Date: 28/04/2022 (MORNING)

Subject with Code: **MA206 LINEAR ALGEBRA**

Marks Obtained	Section A (30)	Section B (20)	Total Marks (50)

INSTRUCTION TO CANDIDATE

1. The booklet (question paper cum answer sheet) consists of two sections. First section consists of MCQs of 30 marks. Candidates may mark the correct answer in the space provided / may also write answers in the answer sheet provided. The Second section of question paper consists of subjective questions of 20 marks. The candidates may write the answers for these questions in the answer sheets provided with the question booklet.
2. The booklet will be distributed to the candidates before 05 minutes of the examination. **Candidates should write their roll no. in each page of the booklet.**
3. Place the Student ID card, Registration Slip and No Dues Clearance (if applicable) on your desk. All the entries on the cover page must be filled at the specified space.
4. Carrying or using of mobile phone / any electronic gadgets (except regular scientific calculator)/chits are strictly prohibited inside the examination hall as it comes under the category of unfair means.
5. No candidate should be allowed to enter the examination hall later than 10 minutes after the commencement of examination. Candidates are not allowed to go out of the examination hall/room during the first 30 minutes and last 10 minutes of the examination.
6. Write on both side of the leaf and use pens with same ink.
7. The medium of examination is English. Answer book written in language other than English is liable to be rejected.
8. All attached sheets such as graph papers, drawing sheets etc. should be properly folded to the size of the answer book and tagged with the answer book by the candidate at least 05 minutes before the end of examination.
9. The door of examination hall will be closed 10 minutes before the end of examination. Do not leave the examination hall until the invigilators instruct you to do so.
10. Always maintain the highest level of integrity. Remember you are a BITian.
11. Candidates need to **submit the question paper cum answer sheets before leaving the examination hall**.

Birla Institute of Technology
Department of Mathematics

SP 2022, End-Semester examination
MA206 : Linear Algebra
Instructor: Subha Sarkar

Date: 28.04.2022
Full Marks: 50
Time: 2 hrs

Instruction:

The question paper has two sections. In section A there are 15 MCQ of 2 marks each. In section B there are 7 subjective type questions. You have to attempt all the questions.

We use the following notations: $M_{n\times n}(\mathbb{R})$ denotes the set of all $n \times n$ real matrices, $\mathcal{P}_n(\mathbb{R})$ denotes the set of all polynomials of degree less than or equal to n, $\mathcal{P}(\mathbb{R})$ denotes the set of all polynomials. For a linear transformation T, $N(T)$ is the kernel space of T, $R(T)$ is the range space of T, $E_\lambda(T)$ is the eigenspace of T corresponding to λ.

Section A

1. Which one of the following is not a subspace of $M_{n\times n}(\mathbb{R})$?
   (a) $\{A \in M_{n\times n}(\mathbb{R}) : A^t = A\}$.
   (b) $\{A \in M_{n\times n}(\mathbb{R}) : A^t = -A\}$.
   (c) $\{A \in M_{n\times n}(\mathbb{R}) : tr(A) = 1\}$.
   (d) $\{A \in M_{n\times n}(\mathbb{R}) : tr(A) = 0\}$.

2. Let $A \in M_{9\times 9}(\mathbb{R})$ be a matrix such that $A^2 = A$. Which of the following is necessarily true?
   (a) A must be the identity matrix.
   (b) A is diagonalizable.
   (c) A must be a zero matrix.
   (d) rank of A is either 0 or 1.

3. Let $A \in M_{4\times 4}(\mathbb{R})$ be a non-zero matrix such that $A^5 = 0$. Which one of the following can be true?
   (a) $A^4 = I$.
   (b) $A^4 = A$.
   (c) $A^4 = 0$.
   (d) $A^4 = -I$.

4. Let V be a vector space of dimension n. Which one of the following statement is false?
   (a) every linearly independent subset of V contains no more than n vectors.
   (b) Any linearly independent subset of V that contains exactly n vectors is a basis.
   (c) each generating set of V contains at least n vectors.
   (d) Any subset of V that contains exactly n vectors is a basis.

5. The sum of eigenvalues of $\begin{pmatrix} -1 & -2 & -1 \\ -2 & 3 & 2 \\ -1 & 2 & -3 \end{pmatrix}$ is
   (a) -3.
   (b) -1.
   (c) 3.
   (d) 1.

6. Let $T : \mathbb{R}^5 \to \mathbb{R}^5$ be such that $T(x_1, x_2, x_3, x_4, x_5) = (21x_1 + 7x_2, -11x_1 + 9x_2, -19x_2 + 35x_3, 15x_2 + 12x_4 + 20x_5, -24x_3 + 21x_4 + 35x_5)$. Then $\operatorname{rank}(T)$ is
   (a) 1.
   (b) 2.
   (c) 3.
   (d) 4.

7. Number of linearly independent eigenvectors of $\begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix}$ is
   (a) 1
   (b) 2
   (c) 3
   (d) 4

8. Which one of the following statement is false?
   (a) There is an one-to-one linear transformation $T : \mathbb{R}^2 \to M_{2\times 2}(\mathbb{R})$.
   (b) There is an onto linear transformation $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^3$.
   (c) There is an one-to-one linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$.
   (d) None of the above.

9. Which of the following set is linearly dependent?
   (a) $\{-x^3 + 2x^2 - 5x, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $\mathcal{P}_3(\mathbb{R})$.
   (b) $\{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}$ in $\mathcal{P}_3(\mathbb{R})$.
   (c) $\{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 0, 0, 1)\}$ in $\mathbb{R}^4$.
   (d) none of the above.

10. Let $u, v \in \mathbb{R}^n$ with the standard inner product. Consider the statements:

P: There exist u and v such that $||u||^2 = 9$, $||v||^2 = 25$, $||u+v||^2 = 4$ and $||u-v||^2 = 16$.

Q: There exist u and v such that $||u|| = 2$, $||v|| = 2$ and $\langle u, v \rangle = 5$.

Then

(a) both P and Q are true.

(b) P is true but Q is false.

(c) Q is true but P is false.

(d) both P and Q are false.

11. Let $T_1, T_2 : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ be such that $T_1(p(x)) = \int_0^x p(t)dt$ and $T_2(p(x)) = p^{'}(x)$. Then

(a) T_1 is one-one but T_2 is not.

(b) T_1 is onto and T_2 is one-one.

(c) T_2 is one-one but T_1 is not.

(d) both T_1 and T_2 are one-one.

12. Let u and v be distinct vectors in $\mathbb{R}^3$. Consider the statements:

P: $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of other.

Q: there exists three linearly dependent vectors in R^3 such that none of the three is a multiple of another.

Then

(a) both P and Q are true.

(b) P is true but Q is false.

(c) Q is true but P is false.

(d) both P and Q are false.

13. Which one of the following is a subspace of $\mathbb{R}^3$ under the usual addition and scalar multiplication?

(a) $\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$.

(b) $\{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$.

(c) $\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$.

(d) $\{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 + 2a_2^2 - 3a_3^2 = 0\}$.

14. Which of the following is not a linear transformation?

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(x_1, x_2) = (x_1, x_2^2)$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(x_1, x_2, x_3) = (x_1 + 2x_2, 5x_3)$.

(c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x_1, x_2, x_3) = (x_1 + 2x_3, 0, x_1 + x_2)$.

(d) none of the above.

15. Let $S_{n\times n}(\mathbb{R})$ be the set of all $n \times n$ skew-symmetric matrices and the characteristics polynomial of each $A \in S_{n\times n}(\mathbb{R})$ is of the form $t^n + a_{n-2}t^{n-2} + a_{n-3}t^{n-3} + \cdots + a_1 t + a_0$. Then the dimension of $S_{n\times n}(\mathbb{R})$ over $\mathbb{R}$ is

    (a) $\frac{n(n-1)-2}{2}$.

    (b) $\frac{n(n-1)}{2}$.

    (c) $\frac{(n-1)^2}{2}$.

    (d) $\frac{(n+2)(n-1)}{2}$.

Section B

16. Find bases for the following subspaces of $\mathbb{R}^5$:

$$W_1 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 - x_2 - x_3 = 0\}$$

and

$$W_2 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = x_2 = x_4 = 0 \text{ and } x_3 + x_5 = 0\}.$$

What are the dimensions of W_1 and W_2?

$\boxed{2}$

17. Give an example of distinct linear transformations $T, U : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = N(U)$ and $R(T) = R(U)$. Justify your answer.

$\boxed{3}$

18. Let V be a vector space and $T : V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$. (T_0 is the zero operator.)

$\boxed{2}$

19. Let V, W and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear. Prove that if $U \circ T$ is one-to-one, then T is one-to-one. Must U also be one-to-one? Justify your answer.

    Can you find linear transformations $S : \mathbb{R}^2 \to \mathcal{P}_2(\mathbb{R})$ and $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^2$ such that $S \circ T = I$? Justify your answer.

$\boxed{3+2=5}$

20. Let V be a finite dimensional vector space and $T : V \to V$ be a linear operator. Let W be an one dimensional T-invariant subspaces of V. Show that $W = E_\lambda(T)$ for some eiganvalue λ of T. (W is said to be T-invariant if $T(W) \subseteq W$).

    Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator $T(x_1, x_2) = (2x_1 - 3x_2, 2x_1 - 2x_2)$. Show that only T-invariant subspaces of $\mathbb{R}^2$ are $\mathbb{R}^2$ and the zero subspace.

$\boxed{2+2=4}$

21. Let $A \in M_{2\times 2}(\mathbb{R})$ be of rank 1. Show that A is either nilpotent or diagonalizable.

$\boxed{2}$

22. Let $A \in M_{11\times 11}(\mathbb{R})$ be a nonzero matrix such that $A^{11} = 0$. Show that A is not diagonalizable.

$\boxed{2}$
