**BIRLA INSTITUTE OF TECHNOLOGY, MESRA, RANCHI**
**(END SEMESTER EXAMINATION MO2022)**

**CLASS: IMSC** **SEMESTER : VII**
**BRANCH: MATHEMATICS & COMPUTING** **SESSION : MO/2022**

**SUBJECT: MA402 ADVANCED COMPLEX ANALYSIS**

**TIME: 03 Hours** **FULL MARKS: 50**

**INSTRUCTIONS:**
**1. The question paper contains 5 questions each of 10 marks and total 50 marks.**
**2. Attempt all questions.**
**3. The missing data, if any, may be assumed suitably.**
***4*. Tables/Data handbook/Graph paper etc., if applicable, will be supplied to the candidates**

---

Q.1(a) Outline how the stereographic projection is used to make the notion of point at infinity tangible in the complex plane. **(CO1, BT2)** [2]

Q.1(b) Applying Cauchy - Goursat theorem for multiple connected domain, compute the value of the integral $I = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz,$ where $C$ is the circle $|z| = 3.$ **(CO1, BT3)** [3]

Q.1(c) Using Cauchy's Integral formula for the $n^{th}$ derivative, derive that: [5]

$$\left| f^n(z_0) \right| \le \frac{n! M}{\rho^n}$$

where $f$ be analytic within and on the circle $C$ defined by $|z - z_0| = \rho$ and $|f(z)| \le M$ (bounded) at each point on $C$. Hence, using the above obtained inequality, also prove that if $f$ becomes analytic and bounded in the entire finite plane, then it is a constant function. **(CO1, BT3)**

Q.2(a) If a bilinear transformation $w = T(z)$ has exactly two fixed points $z_1$ and $z_2$, then for some non - zero constant $k$, show that they satisfy the equation [5]

$$\frac{w - z_1}{w - z_2} = k \frac{z - z_1}{z - z_2}$$

**(CO2, BT2)**

Q.2(b) Develop the Laurent series expansions of the function $f(z) = \frac{1}{(z-1)(z-2)}$ in the following regions: [5]

i. $1 < |z| < 2$

ii. $0 < |z - 2| < 1$

**(CO2, BT3)**

Q.3(a) If $z = z_0$ is some singularity of the given function $f(z)$. Then, under what conditions it is called to be non - isolated one. Is it possible to expand the function $f(z)$ as Laurent series if $z_0$ is a non - isolated singularity? Explain with proper reasoning. **(CO2, BT2)** [2]

Q.3(b) Describe how the function $f(z) = z^{\frac{1}{2}}$ is a multivalued function, mentioning about the branch, branch point and branch cut for it. [3]

**(CO2, BT3)**

Q.3(c) Determine the residues at each of the singularities of the function $f(z) = \frac{z^2}{(z^2 + 3z + 2)^2}$. Hence, using these residues, evaluate the integral $\oint_C f(z) dz,$ when $C$ is an ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1.$ [5]

**(CO3, BT3)**

**PTO**

Q.4(a) If a function $f(z)$ is meromorphic inside a simple closed contour $C$ and $f(z)$ is analytic and has no zeros on $C$, then prove that: [5]

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N_z - N_p \qquad (*)$$

where $N_z$ is the number of zeros and $N_p$ is the number of poles inside $C$ (a pole or zero of order $m$ must be counted $m$ times).

In particular, if $f(z)=\frac{(z-2)^4}{z^3(z-1)^4}$, then using expression (*), obtain the value of integral $\oint_C \frac{f'(z)}{f(z)}dz$, where $C$ is the circle $|z|=3$. Also, if possible, determine the winding number of the transformation $w=f(z)$ from the $z-$plane to $w-$plane.

**(CO2, BT2)**

Q.4(b) State Rouche's theorem. Hence, using it, find the number of the roots of the polynomial [5]

$$P(z)=z^9-2z^6+z^2-8z-2$$

that lie inside the circle $|z| = 1.$ **(CO3, BT3)**

Q.5(a) Define Weierstrass primary factor of order $E_p(z)$ of order $p$. Demonstrate how the Weierstrass primary factor $E_p(z)$ becomes convergent for large values of $p$. [5]

**(CO2, BT2)**

Q.5(b) The following information about an entire function $f(z)$ is given [5]

i) the only zeros of the function are at $z=n^2$, where $n \in \mathrm{N}$ (set of natural numbers)
ii) genus of the canonical product associated with the function is one, whereas the genus of the function itself is two.

With the above information, is it possible to construct the entire function in Weierstrass Factorized form. If so, then find the suitable form (s) of $f(z)$.

**(CO3, BT3)**

**:::::25/11/2022:::::E**