Solution: Newton's method produces the following sequences of values for x_1, x_2 and $[f(x^{k+1}) - f(x^k)]$ you should try to verify the calculation shown in the following table the trajectory is traced

Iteration	x_1	x_2	$f(x^{k+1}) - f(x^k)$
0	1.00000	1.00000	6.0000
1	1.00000	−0.50000	1.5
2	1.391304	−0.695652	4.09×10^{-1}
3	1.745944	−0.948798	6.49×10^{-2}
4	1.986278	−1.048208	2.53×10^{-3}
5	1.998734	−1.000170	1.63×10^{-6}
6	1.9999996	−1.0000002	2.75×10^{-12}

You can calculate between iterations 2 and 3 that $c = 0.55$, and between 3 and 4 that $c = 0.74$. Hence quadratic convergence can be demonstrated numerically.

This module (Module 7) treats more difficult problem involving minimization or maximization of a nonlinear objective function subject to linear or non linear constraints

* Direct substitution
Minimize $f(x) = 4x_1^2 + 5x_2^2$
Subject to $2x_1 + 3x_2 = 6$

Minimize $f(x)$ $\quad x = [x_1, x_2 \; \dots \; x_n]^T$

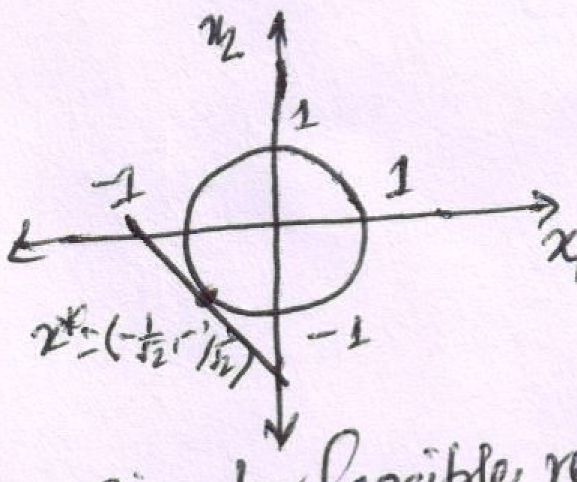
Subject to: $h_i(x) = b_i \quad i = 1, 2, \dots m$

$g_j(x) \le g_j \quad j = 1, \dots n$

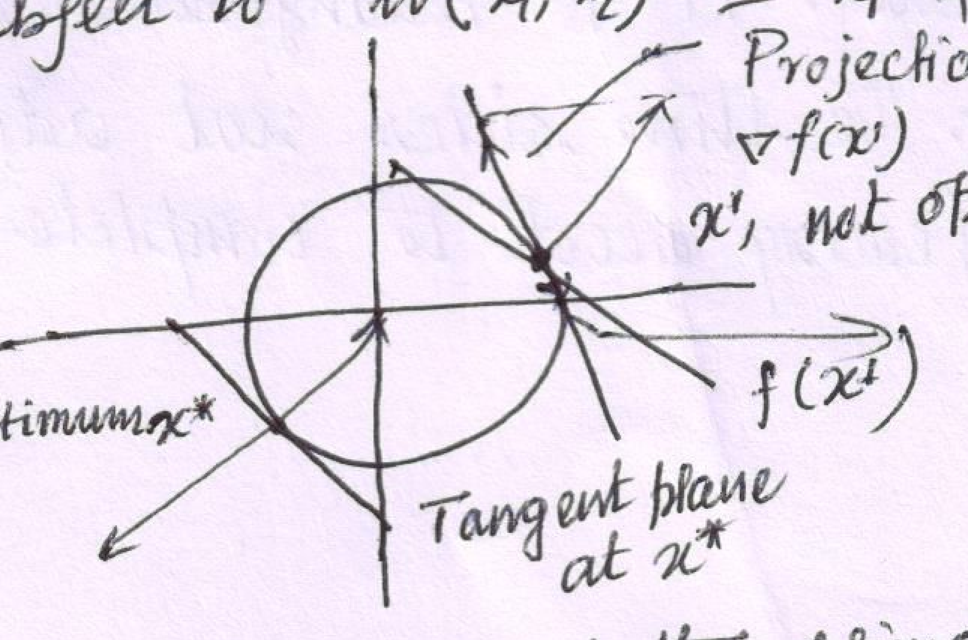
The inequality constraints can be transformed into equality constraints as explained, so we focus first on problems involving only equality constraints.

Graphic interpretation of a constrained optimization problem.

Minimize $f(x_1, x_2) = x_1 + x_2$, subject to $h(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$



Circular feasible region with objective function contours and the constraints



Gradients at the optimal point and at a non optimal point.

- Problem containing only equality constraints -

A general equality constrained NLP with m constraints and n variables can be written as

$$\text{Maximize } f(x) \qquad x = [x_1, \dots x_n]$$

$$\text{Subject to } h_j(x) = b_j \qquad j = 1, 2, \dots m$$

Continuous first partial derivatives - define a Lagrange multiplier λ_j and let corresponding to each constraint $h_j = b_j$. Let $\lambda = (\lambda_1, \dots \lambda_m)$ be the vector of these multipliers. The Lagrangian function for the problem is $L(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j [h_j(x) - b_j]$

and the first order necessary conditions are

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial h_j}{\partial x_i} = 0 \qquad i = 1, \dots n$$

$$h_j(x) = b_j \qquad j = 1, \dots m$$

* Minimize $f(x) = 4x_1^2 + 5x_2^2$

Subject to: $h(x) = 0 = 2x_1 + 3x_2 - 6$

Let $L(x,\lambda) = f(x) + \lambda(h(x)) = 4x_1^2 + 5x_2^2 + \lambda(2x_1 + 3x_2 - 6)$

Apply the necessary conditions

$$\frac{\partial L(x,\lambda)}{\partial x_1} = 8x_1 + 2\lambda = 0 \Rightarrow x_1 = -\lambda/4$$

$$\frac{\partial L(x,\lambda)}{\partial x_2} = 10x_2 + 3\lambda = 0 \Rightarrow x_2 = -\frac{3}{10}\lambda$$

$$\frac{\partial L(x,\lambda)}{\partial \lambda} = 2x_1 + 3x_2 - 6 = 0$$

$$2\left(-\frac{\lambda}{4}\right) + 3\left(-\frac{3}{10}\lambda\right) - 6 = 0$$

$$\lambda = -6/1.4 = -4.286$$

$$x_1^* = 1.071, \quad x_2^* = 1.286$$

* Minimize $x_1 + x_2$

Subject to $x_1^2 + x_2^2 = b$

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = b$$

$$\frac{1}{2\lambda^2} = b$$

$$\frac{1}{4\lambda^2} = \frac{b}{2}$$

$1 + 2\lambda x_1 = 0 \qquad x_1 = -\frac{1}{2\lambda}$

$1 + 2\lambda x_2 = 0 \qquad x_2 = -\frac{1}{2\lambda}$

$x_1^2 + x_2^2 - b = 0 \qquad \frac{1}{2\lambda^2} = b \Rightarrow \frac{1}{\lambda^2} = 2b \Rightarrow \frac{1}{4\lambda^2} = 2b/2$

$$\frac{1}{2\lambda} = \pm\sqrt{2b/2}$$

$$x_2 = x_1 = -\frac{1}{2\lambda} = -\sqrt{2b/2} \qquad \lambda^* = (2b)^{-1/2}$$

The minimal objective function value, sometimes called the optimal value function is

$$V(b) = -(2b)^{1/2}$$

The highway department is planning to build a picnic area for motorist along a major highway. It is rectangular with area of 5000 square yards. It is to be fenced of on the three sides not adjacent to the highway. what is the least amount of fencing need to complete the job.

— Problem Containing only Inequality Constraints —

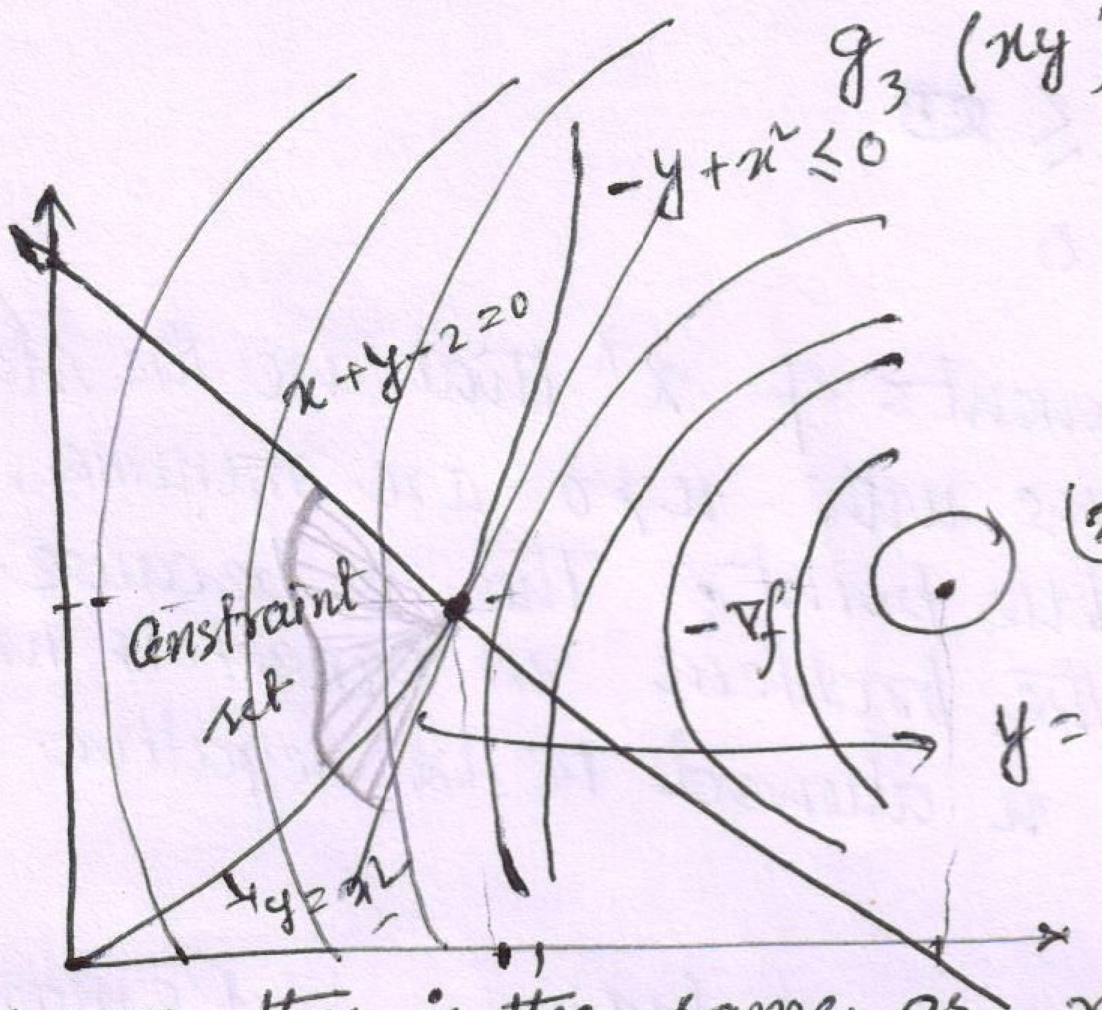
The first order necessary conditions for problem with inequality constraints are called the Kuhn-Tucker conditions (also called Karush-Kuhn-Tucker conditions)

$$\text{Minimize } f(x,y) = (x-2)^2 + (y-1)^2$$

$$\text{Subject to } \quad g_1(x,y) = -y + x^2 \leq 0$$

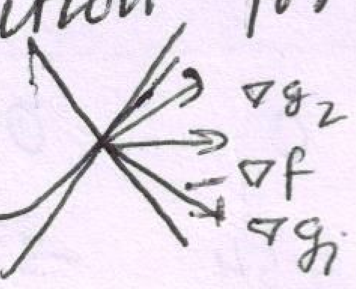
$$g_2(x,y) = x + y \leq 2$$

$$g_3(x,y) = y \geq 0$$



Geometry of a constrained optimization problem. The feasible region lies within the binding constraints plus the boundaries themselves.

Neither case can occur at an optimal point, and both cases are excluded if and only if $-\nabla f$ lies within the cone generated by ∇g_1 and ∇g_2. Of course, this is the same as requiring that ∇f lie within the cone generated by $-\nabla g_1$ and $-\nabla g_2$. This leads to the usual statement of KTC, that is if f and all g_i are differentiable, a necessary condition for a point x^* to be a constrained minimum of the problem



Gradient of objective contained in convex cone.

$$\text{Minimize } f(x)$$

$$\text{Subject to } \quad g_j(x) \leq g \qquad j = 1, \ldots n$$

is that at x^* ∇f lies within the cone generated by the negative gradients of the binding constraints.

$$\nabla f(x^*) = \sum_{j\in I} u_j^* [-\nabla g_j(x^*)]$$

$$\text{where } \quad u_j^* \geq 0 \qquad j \in I$$

and I is the set of indices of the binding inequality constraints. The multipliers u_j^* are analogous to λ_i defined for equality constraints.

$$\nabla f(x^*) + \sum_{j=1}^{n} u_j^* \nabla g_j(x^*) = 0$$

- Complementary
- Slackness

$$u_j^* \geq 0, \quad u_j^* [g_j(x^*) - g] = 0$$

$$g_j(x^*) \leq g, \quad j = 1 \ldots n$$

From the lagrangian $L(x,u) = f(x) + \sum_{j=1}^{n} u_j [g_j(x) - g]$

where the u_j are viewed as lagrange multipliers for the inequality constraints $g_j(x) \leq g$

$$\text{Minimize } f(x), \quad \text{subject to } \quad h_i(x) = b_i \qquad i = 1, \ldots, m$$

$$\text{and } \quad g_j(x) \leq g \qquad j = 1 \ldots n$$

Define lagrange multipliers λ_i for equalities and u_j for the inequalities and form the lagrangian function

$$L(x, \lambda, u) = f(x) + \sum_{i=1}^{m} \lambda_i [h_i(x) - b_i] + \sum_{j=1}^{n} u_j [g_j(x) - g]$$

$$\nabla_x L(x^*, u^*, \lambda^*) = \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) + \sum_{j=1}^{n} u_j^* \nabla g_j(x^*) = 0$$

and complementary slackness hold for the inequalities

$$u_j^* \geq 0 \quad u_j^* [g_j(x^*) - g] = 0 \quad j = 1, \ldots n$$

What is optimization?

Optimization is essentially about finding the best solution to a given problem from a set of feasible solutions. It consist of three components

- the objective or objectives, that is, what do we want to optimize?
- a solution (decision) vector, that is, how can we achive the optimal objective?
- the set of all feasible solutions, that is, among which possible option may we choose to optimize?

Examples

- Airline companies schedule crews and aircraft to minimize their cost
- Investors create portfolios to avoid the risks and achieve the maximum profits
- Manufactures minimize the production costs and maximize the efficiency
- Bidders optimize their bidding strategies to achieve best results
- Physical system tends to a state of minimum energy

What are necessary and sufficient conditions for a local minimum?

- Necessary conditions: conditions satisfied by every local minimum
- Sufficient conditions: conditions which guarantee a local minimum

Easy to characterize a local minimum if f is sufficiently smooth

Formulation of optimization Problems: Degree of freedom, Objective function, Constraints, Continuity of function, unimodal function and multimodal func. Concave and convex function

Stationary points

Let $f: R \to R$, $f \in C^1$

Consider the problem, $\min_{x \in R} f(x)$

Definition

x^* is called a stationary point if $f'(x^*) = 0$

Necessity of an algorithm

- Consider the problem

$$\min_{x \in R} (x-2)^2$$

we first find the stationary points (which satisfy $f'(x) = 0$)

$$f'(x) = 0 \Rightarrow 2(x-2) = 0 \Rightarrow x^* = 2$$

$f''(2) = 2 > 0 \Rightarrow x^*$ is a strict local minimum

- Stationary points are found by solving a nonlinear equation

$$g(x) = f'(x) = 0$$

Finding the real roots of $g(x)$ may not be always easy

- Consider the problem to minimize $f(x) = x^2 + e^x$
- $g(x) = 2x + e^x$
- Need an algorithm to find x which satisfies $g(x) = 0$