Simplex Method or Simplex Algorithm

After the introduction of the slack and surplus variables and by proper adjustment of z, let a LPP be optimize $z = cx$

Subject to $Ax = b$, $x \geq 0$ $[A]_{m \times n}$

where $c = (c_1, c_2, \cdots, c_r, \overbrace{0, 0, 0 \cdots 0}^{n-r})$ an n component row vector

$x = [x_1, x_2, \cdots x_r, x_{r+1}, \cdots x_n]$ an n component column vector

The components $x_{r+1}, x_{r+2}, \cdots x_n$ are either slack or surplus variables. We make an assumption $(m < n)$ [m the number of constraints] and for this make an assumption that all components of b are non negative by proper adjustment $A = (a_1, a_2, \cdots a_n)$ where a_j is the j^{th} column vector of the co-efficient matrix which are called the activity vectors associated with the variable x_j $[j = 1, 2, \cdots n]$

Let $\beta_1, \beta_2, \beta_3 \cdots \beta_m$ be a set of m linearly independent column vectors taken from $a_1, a_2, \cdots a_n \in A$. Then one basis matrix B is given by

$$B = B(\beta_1, \beta_2, \beta \cdots \beta_m)$$

Let $x_{B_1}, x_{B_2}, \cdots x_{B_m}$ be the basic variables associated with the column vectors $\beta_1, \beta_2, \cdots \beta_m$ respectively. Then the basic variable vector is

$$x_B = [x_{B_1}, x_{B_2}, \cdots x_{B_m}]$$

The solution set corresponding to the basic variables is

$$\hat{x}_B \text{ or simply } x_B = B^{-1} b$$

We assume that $x_B \geq 0$ i.e. the solution is a BFS and in that case the basis is called an admissible basis in the simplex theory

Let $c_{B_1}, c_{B_2}, \cdots c_{B_m}$ be the coefficients of $x_{B_1}, x_{B_2}, \cdots x_{B_m}$ respectively in the objective function $z = c.x$ then $c_B = (c_{B_1}, c_{B_2}, \cdots c_{B_m})$

Now a value of z_B is defined as

$$z_B = c_{B_1} x_{B_1} + c_{B_2} x_{B_2} + \cdots + c_{B_m} x_{B_m} = c_B x_B$$

Now as $(\beta_1, \beta_2, \cdots \beta_m)$ are linearly independent, then all the column vectors a_j can be expressed as the linear combination of $\beta_1, \beta_2, \cdots \beta_m$

Let $a_j = \beta_1 y_{1j} + \beta_2 y_{2j} + \cdots + \beta_m y_{mj} = B y_j$

the inner product of B and y_j is an m-component column vector given by $y_j = [y_{1j}, y_{2j}, \cdots y_{mj}]$

$$y_j = B^{-1} a_j$$

Net evaluation: Evaluation is the inner product of the row vector c_B and the column vector y_j which is usually denoted by z_j and z_j is given by

$$z_j = c_B y_j = c_B B^{-1} a_j = c_{B_1} y_{1j} + c_{B_2} y_{2j} + \cdots + c_{B_m} y_{mj}$$

and $z_j - c_j$ is called as net evaluation

The function $z_j - c_j$ $[j = 1, 2, \cdots n]$ plays a very important role in determining the optimal stage in the case of solving LPP by the simplex method.

Search for a basis which will give a B.F.S.

$c = (c_1, c_2, c_j - c_n) \to$ a row vector

$x = [x_1, x_2, \cdots x_j \cdots x_m] \to$ a column vector

$a_j = [a_{1j}, a_{2j}, \cdots a_{ij} \cdots a_{mj}] \to$ a column vector

$b = [b_1, b_2 \cdots b_j \cdots b_m] \to$ a column vector

out of the vectors $a_1, a_2, \cdots a_j, \cdots a_n$ we shall have to select arbitrarily m vectors which are linearly independent (there exists always at least one set of such vectors, since the equations are linearly independent) which form a basis matrix B. With that basis, find out the basic solution and let us assume that the BS $x_B = B^{-1}b \geq 0$ ie the solution is a B.F.S. Such basis is called an admissible basis to start the simplex method.

In all practical problems, there always exists an identity matrix I_m and $b \geq 0$ and $x_B = I_m^{-1} b = I_m b = b \geq 0$
so it will not be difficult to find and initial BFS.

Let the BFS $\quad x_B = [x_{B1}, x_{B2}, \cdots x_{Bm}]$

$c_B = (c_{B1}, c_{B2}, \cdots c_{Bm})$

an m component row vector

Now we calculate the column vector $y_1, y_2, y_j \cdots y_n$ where $y_j = [y_{1j}, y_{2j}, y_{ij} \cdots y_{mj}]$

By using the formula $\quad y_j = B^{-1} a_j \quad [j = 1, 2, \cdots n]$

Since in all practical purposes the initial basis is I_m thus $y_j = I_m^{-1} a_j = I_m a_j = a_j$

Thus initially $\quad y_{ij} = a_{ij}$

Now we find out $\quad z_j = c_B y_j = c_B B^{-1} a_j$

thus $z_j = c_{B1} y_{1j} + c_{B2} y_{2j} + c_{Bi} y_{ij} + c_{Bm} y_{mj}$

and the value of the objective function corresponding to the basis B which is denoted by $z_B = z_0$ given by

$z_0 = c_{B1} x_{B1} + c_{B2} x_{B2} + c_{Bi} x_{Bi} + \cdots + c_{Bm} x_{Bm}$

Now we calculate all $z_j - c_j$, $[j = 1, 2, \cdots n]$

-optimality test-

For a minimization problem if at any stage all $z_j - c_j \geq 0$ $[j = 1, 2, \cdots n]$ the problem is at the optimal stage. If at least one $z_j - c_j < 0$ then the problem is not at the optimal stage and we shall have to proceed further. If at least one $z_j - c_j < 0$ and at least one $y_{ij} > 0$, then the value of the objective function can be improved further or at least remains same. If any $z_j - c_j < 0$ and all $y_{ij} \leq 0$ $[i = 1, 2, \cdots m]$ the problem has no finite optimal value and the problem is said to have an unbound solution. [Actually all the data are to be placed in a table and the format of the simplex table will be placed in some later stage]

Step 8 If none of these two criteria be satisfied, then choose the minimum most value from among all the $(z_j - c_j)$. This minimum most value of $(z_j - c_j)$ must be there as we have checked for step 6 earlier. Suppose the minimum most value of $(z_j - c_j)$ occurs for $j = k$ then a_k will be the entering vector in the new basis, that is to be formed. Fix up this vector with $(\uparrow)$ below the column of the entering vector called the key column.

If all $y_{ik} < 0$ then the solution will be unbounded and here leave the problem. If at least one $y_{ik} > 0$ then proceed to step 9. If again the minimum most $(z_j - c_j)$ be not unique, then any one of the vectors associated with this same minimum value of $(z_j - c_j)$ may be taken to be the entering vector.

Step 9 To figure out the departing vector, that is the vector that is to removed from the current basis compute $\text{Min}\left\{ \frac{x_{Bi}}{y_{ik}}, y_{ik} > 0 \right\}$ where k is the arrow indicated column. If this minimum occurs for one and only one value of i say $i = r$, then the vector β_r will have to be removed from the basis and β_r will be the departing vector. This r^{th} row is called the key row. We mark this vector by $(\downarrow)$ below the column of the departing vector. In the next basic feasible solution, this x_{B_r} will be zero and x_{BK} will be non-zero.

If on the other hand $\text{Min}\left\{ \frac{x_{Bi}}{y_{ik}}, y_{ik} > 0 \right\}$ occurs for more than one variable will vanish in the next solution generating degeneracy of the BFS. We shall discuss in a separate chapter how this degeneracy can be resolved.

Step 10 Then write c_k, a_k and x_k respectively in place of c_r, β_r and x_r in the r^{th} row of the columns headed by c_B, B and x_B respectively.

Step 11 The intersection of the key row and the key column, that is y_{rk} is called the key number or the pivot element. Divide the key row elements of the current tableau by y_{rk} and this will be the r^{th} row of the new tableau. The (r,k) th element of the new tableau obviously will be 1.

As is evident, the key element will be a positive number.

Step 12 The other rows of the new tableau are then computed as follows. Substract the y_{1k} time the r^{th} row of the new tableau from the first row elements of the old tableau. Subtract the y_{2k} times the r^{th} row elements of the new tableau from the second row elements of the old tableau. Apply similar operation for all the rows.

Notice that the new vector obtained in the key column has been reduced to a unit vector e_r.

Step 13: Repeat step 6 and step 7. Then if required, proceed with the step 8, 9 and 10 successively.

Step 14 Use simplex iteration to remove all the artificial vectors from the basis if possible, so that the new basis contains only the original vectors and the vectors corresponding to slack and surplus variables called the legitimate vectors.

Solve the following L.P.P.

Maximize $z = 60x_1 + 50x_2$

Subject to $x_1 + 2x_2 \leq 40$

$3x_1 + 2x_2 \leq 60$

and $x_1, x_2 \geq 0$

Here both the constraints are $(\leq)$ type, hence introducing slack variables x_3 and x_4, we rewrite the problem in the standard form as

$$\text{Maximize } z = 60x_1 + 50x_2 + 0\cdot x_3 + 0\cdot x_4$$

$$\text{Subject } x_1 + 2x_2 + x_3 + 0\cdot x_4 = 40$$

$$3x_1 + 2x_2 + 0\cdot x_3 + x_4 = 60$$

$$x_1, x_2, x_3, x_4 \geq 0$$

The initial basis matrix is the identity matrix given by the co-efficients of x_3 and x_4 and as such they will form the basic solution

Here $(c_1, c_2, c_3, c_4) = (60, 50, 0, 0)$

$a_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $a_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $a_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $b = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$

We see that the vectors a_3 and a_4 form the initial basis and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = B^{-1}$

$x_B = B^{-1}b = I^{-1}b = b \quad [\because B = I]$

Thus $[x_{B_1}, x_{B_2}] = [40, 60]$ Also $c_B = (c_{B_1}, c_{B_2}) = (0, 0)$

$\min\left(\frac{x_{Bi}}{y_{ik}}, y_{ik} > 0\right)$

TABLEAU I

			c_j	60	40	0	0	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	
0	a_3	x_3	40	1	2	1	0	40/1
0	a_4	x_4	60	3	2	0	1	60/3
			$z_j - c_j$	-60	-40	0	0	

$z_j - c_j < 0$ hence this tableau will not give the optimal solution

Now index number $= \sum$ numbers in each column $\times$ corresponding number in the c_B column(c_B) $-$ number in the objective row

Minimum most number in the index row is (-60) and hence the column corresponding to this number is the key column and a_1 is thus the entering vector

Here $\min\{40, 20\} = 20$

The second being the smallest, second row is the key row and a_4 under B is the departing vector. Then 3 being at the intersection of key column and key row is the key number.

			c_j	60	50	0	0	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	
0	a_3	x_3	20	0	$1\frac{1}{3}$	1	$-1/3$	$20/1\frac{1}{3}$
60	a_1	x_1	20	1	$2/3$	0	$1/3$	$20/2/3$
			$z_j - c_j$	0	-10	0	20	
50	a_2	x_2	15	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	
60	a_1	x_1	10	1	0	$-1/2$	$1/2$	
			$z_j - c_j$	0	0	$\frac{15}{2}$	$35/2$	

* Maximize $z = 5x_1 + 2x_2$

Subject to $6x_1 + 10x_2 \leq 30$

$10x_1 + 4x_2 \leq 20$

$x_1, x_2 \geq 0$

$z_j - c_j \geq 0$ Hence this tableau gives the optimal solution. optimal solution is thus $x_1 = 10$, $x_2 = 15$ and $z_{max} = 1350$

Simplex Method

Solve the following LPP by simplex method

Maximize, $Z = 4x_1 + 7x_2$, subject to $2x_1 + x_2 \leq 1000$, $x_1 + x_2 \leq 600$, $-x_1 - 2x_2 \geq -1000$, $x_1, x_2 \geq 0$

solⁿ: This is a maximization problem

Multiplying the third constraints by (-1) we get $x_1 + 2x_2 \leq 1000$

Hence all $b_i \geq 0$ and all constraints are attached with sign '≤' type. Introducing three slack variables x_3, x_4, x_5 one to each constraint we get the following converted equations

$$2x_1 + x_2 + x_3 = 1000$$
$$x_1 + x_2 + x_4 = 600$$
$$x_1 + 2x_2 + x_5 = 1000$$

The adjusted objective function z is given by

$$z = 4x_1 + 7x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5$$
$$c = (4, 7, 0, 0, 0)$$

$$a_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad a_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad a_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad a_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1000 \\ 600 \\ 1000 \end{bmatrix} \geq 0$$

The unit basis $B = (a_3, a_4, a_5) = I_3$ and taking this as initial basis we have

Initial BS $x_B = B^{-1} b = I_3^{-1} b = I_3 b = b \geq 0$ which is feasible

Then B is the admissible basis and

$$x_B = [x_{B1}, x_{B2}, x_{B3}] = [x_3, x_4, x_5] = [b_1, b_2, b_3] = [1000, 600, 1000]$$

$$c_B = (c_{B1}, c_{B2}, c_{B3}) = (c_3, c_4, c_5) = (0, 0, 0) = 0$$

$$z_0 = c_B x_B = 0 x_B = 0$$

$$y_j = B^{-1} a_j = I_3 a_j = a_j \; [j = 1, 2, 3, 4, 5]$$

$$z_j - c_j = c_B y_j - c_j = 0 y_j - c_j = -c_j \; [j = 1, 2, 3, 4, 5]$$

with the data given below/above we can construct the initial simplex table

Initial simplex table:

		c_j	4	7	0	0	0	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	Min. Ratio
0	a_3	x_3	1000	2	1	1	0	0	1000/1 = 1000
0	a_4	x_4	600	1	1	0	1	0	600/1 = 600
0	a_5	x_5	1000	1	2	0	0	1	1000/2 = 500
$z_j - c_j$ =				-4	-7 ↑	0	0	0 ↓	

		c_j	4	7	0	0	0	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	Min Ratio
0	a_3	x_3	500	3/2	0	1	0	-1/2	$500 \times \frac{2}{3} = \frac{1000}{3}$
0	a_4	x_4	100	1/2*	0	0	1	-1/2	$100 \times \frac{2}{1} = 200$
7	a_2	x_2	500	$\frac{1}{2}$	1	0	0	1/2	$500 \times \frac{2}{1} = 1000$
$z_j - c_j$				-1/2 ↑	0	0	0 ↓	7/2	

		c_j	4	7	0	0	0	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	Min, Ratio
0	a_3	x_3	200	0	0	1	-3	1	
4	a_1	x_1	200	1	0	0	2	-1	
7	a_2	x_2	400	0	1	0	-1	1	
$z_j - c_j$				0	0	0	1	3	

Solution to the problem when some artificial variables are added to get a unit basis in the co-efficient matrix:—

In this section, we will present a generalized version of the simplex method that will solve both maximization and minimization problems with any combination of $\leq$ $\geq$ $=$ constraints

Maximize $z = 2x_1 + x_2$ subject to $x_1 + x_2 \leq 10$, $-x_1 + x_2 \geq 2$, $x_1, x_2 \geq 0$

To form an equation out of the first inequality, we introduce a slack variable x_3 as before and write $x_2 + x_2 + x_3 = 10$

To form an equation out of the second inequality we introduce a second variable x_4 and subtract it from the left side so that we can write

$$-x_1 + x_2 - x_4 = 2$$

The variable x_4 is called a surplus variable, because it is the amount (surplus) by which the left side of the inequality exceeds the right side

We now express the linear programming problem as a system of equations

$$\begin{aligned} x_1 + x_2 + x_3 \quad &= 10 \\ -x_1 + x_2 \quad - x_4 &= 2 \end{aligned} \qquad x_1, x_2, x_3, x_4 \geq 0$$

It can be shown that a basic solution of a system is not feasible if any of the variables (excluding p) are negative. Thus a surplus variable is required to satisfy the nonnegative constraint

An initial basic solution is found by setting the nonbasic variables x_1 and x_2 equal to 0. That is $x_1 = 0$, $x_2 = 0$, $x_3 = 10$, $x_4 = -2$, $p = 0$, This solution is not feasible because the surplus variable x_4 is negative.

In order to use the simplex method on problems with mixed constraints, we turn to a device called an artificial variable. This variable has no physical meaning in the original problem and is introduced solely for the purpose of obtaining a basic feasible solution so that we can apply the simplex method

An artificial variable is a variable introduced into each equation that has a surplus variable. To ensure that we consider only basic feasible solutions an artificial variable is required to satisfy the non-negative constraint.

Returning to our example, we introduce an artificial variable x_5 into the equation involving surplus variables x_4 $\rightarrow$ $x_1 + x_2 - x_4 + x_5 = 2$

To prevent an artificial variable from becoming part of an optimal solution to the original problem, a very large 'penalty' is introduced into the objective function. This penalty is created by choosing a positive constant M so large that the artificial variable is forced to be 0 in any final optimal solution of the original problem

We then add the term $-Mx_5$ to the objective function $z = 2x_1 + x_2 - Mx_5$

We now have a new problem, called the modified Problem:

Maximize $z = 2x_1 + x_2 - Mx_5$ subject to $x_1 + x_2 + x_3 = 10$, $x_1 + x_2 - x_4 + x_5 = 20$

Big M method: From the Modified Problem

- If any problem constraints have negative constraints on the right hand side multiply both sides by -1 to obtain a constraint with a nonnegative constant. Remember to reverse the direction of the inequality if the constraint is an inequality.
- Introduce a slack variable for each constraint of the form ≤ 0

Problem having no feasible solution

Example:- Solve the LPP Maximize $Z = 2x_1 - 3x_2$ subject to

$$2x_1 + x_2 \leq 8 \qquad x_1, x_2 \geq 0$$
$$10x_1 + 11x_2 \geq 100$$

Introducing slack variable x_3 and surplus variable x_4 one to each of the constraints respectively we get the following converted equations

$$2x_1 + x_2 + x_3 = 8$$
$$10x_1 + 11x_2 - x_4 = 100$$

The coefficient matrix does not contain a unit basis. To get a unit basis one artificial variable x_5 is to be added to the LHS of the 2nd equation and the set of equations are

$$2x_1 + x_2 + x_3 = 8$$
$$10x_1 + 11x_2 - x_4 + x_5 = 100$$

$Z = 2x_1 - 3x_2 + 0x_3 + 0x_4 - Mx_5$ [Assigning very large -ve price to the artificial variable x_5]

The vectors a_3 and a_5 contribute a unit basis. Initial solution

$$x_B = [x_3, x_5] = [8, 100]$$
$$c_B = (c_3, c_5) = (0, -M) \text{ and } y_j = B^{-1} a_j = a_j$$
$$Z = c_B x_B = -100M$$

Simplex Tables

			y	2	-3	0	0	-M	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	Min Ratio
0	a_3	x_3	8	2	1*	1	0	0	$\frac{8}{1} = 8 \rightarrow$
-M	a_5	x_5	100	10	11	0	-1	1	$\frac{100}{11} = 9\frac{1}{11}$
	$z_j - y_j$			$-10M-2$	$-11M+3$ ↑	0	M	0	
-3	a_2	x_2	8	2	1	1	0	0	
-M	a_5	x_5	2	-12	0	-11	-1	1	
	$z_j - y_j$			$12M-4$	0	$11M-3$	M	0	

verify the result by Geometrical / Graphical method

The optimality conditions have been satisfied. But the artificial vector as in the basis at positive level. Hence the only conclusion is that the problem has no FS in this case. There is no need to calculate the value of the objective function at the final stage.

Example: Solving by Big M-method prove that the following LPP has no feasible solution

Maximize $Z = 2x_1 - x_2 + 5x_3$

Subject to
$$\left.\begin{aligned} x_1 + 2x_2 + 2x_3 &\leq 2 \\ \tfrac{5}{2}x_1 + 3x_2 + 4x_3 &= 12 \\ 4x_1 + 3x_2 + 2x_3 &\geq 24 \end{aligned}\right\} x_1, x_2, x_3 \geq 0$$

$$x_1 + 2x_2 + 2x_3 + x_4 = 2$$
$$\tfrac{5}{2}x_1 + 3x_2 + 4x_3 + x_5 = 12 \qquad x_5 \geq 0$$
$$4x_1 + 3x_2 + 2x_3 - x_6 + x_7 = 24$$

Thus $Z = 2x_1 - x_2 + 5x_3 + 0x_4 - Mx_5 + 0x_6 - Mx_7$

Simplex Tables

		c_j		2	−1	5	0	−M	0	−M	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	Min Ratio
0	a_4	x_4	2	1	2	2	1	0	0	0	$2/1 = 2 \rightarrow$
−M	a_5	x_5	12	5/2	3	4	0	1	0	0	$12 \times \frac{2}{5} = \frac{24}{5}$
−M	a_7	x_7	24	4	3	2	1	0	−1	1	$\frac{24}{4} = 6$
$z_j - c_j$				$-\frac{13M}{2}-2$ ↑	−6M+1	−6M−5	0 ↓	0	M	0	
2	a_1	x_1	2	1	2	2	1	0	0	0	
−M	a_5	x_5	7	0	−2	−1	−5/2	1	0	0	
−M	a_7	x_7	16	0	−5	−6	−3	0	−1	1	
$z_j - c_j$				0	7M+5	7M−1	$\frac{11}{2}M+2$	2	M	0	

All $z_j - c_j \geq 0$ $j = 1, 2, \ldots 7$ in second table. Thus we need not complete the second table. Two artificial variables x_5 and x_7 are present at the positive level in the optimal solution. Then the only conclusion is that the problem has no feasible solution.

* Show that the following linear programming problem has no feasible solution

Maximize $z = x_1 + 4x_2 + 3x_3$

subject to
$$\left.\begin{array}{l} 2x_1 - x_2 + 5x_3 = 40 \\ x_1 + 2x_2 - 3x_3 \geq 22 \\ 3x_1 + x_2 + 2x_3 = 30 \end{array}\right\} x_1, x_2, x_3 \geq 0$$

Simplex Tables

		c_j		1	4	3	−M	0	−M	−M	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	Min Ratio
−M	a_4	x_4	40	2	−1	5	1	0	0	0	40/2 = 20
−M	a_6	x_6	22	1	2	−3	0	−1	1	0	22/1 = 22
−M	a_7	x_7	30	3*	2	2	0	0	0	1	$30/3 = 0 \rightarrow$
$z_j - c_j$				−6M−1 ↑	−3M−4	−4M−3	0	M	0	0 ↓	
−M	a_4	x_4	20	0	−7/3	11/3	1	0	0	−2/3	
−M	a_6	x_6	12	0	4/3	−11/2	0	−1	1	1/3	
1	a_1	x_1	10	1	$\frac{2}{3}$	$\frac{2}{3}$	0	0	0	1/3	
$z_j - c_j$				0	$M - \frac{10}{3}$	$\frac{11}{6}M - \frac{7}{3}$	0	M	0	+ve	

Two artificial variables x_4 and x_6 are at positive level in the basis in the optimal solution. Then the only conclusion is that the problem has no feasible solution.